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Semi-linear degenerate backward stochastic partial differential equations and associated forward–backward stochastic differential equations☆

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Abstract

In this paper, we consider the Cauchy problem of semi-linear degenerate backward stochastic partial differential equations (BSPDEs) under general settings without technical assumptions on the coefficients. For the solution of semi-linear degenerate BSPDE, we first give a proof for its existence and uniqueness, as well as regularity. Then the connection between semi-linear degenerate BSPDEs and forward–backward stochastic differential equations (FBSDEs) is established, which can be regarded as an extension of the Feynman–Kac formula to the non-Markovian framework.

Keywords: Backward stochastic partial differential equations; Semi-linear degenerate equations; Forward–backward stochastic differential equations; Feynman–Kac formula

1. Introduction

BSPDEs were introduced by Bensoussan [2,3] as the adjoint equation of SPDE control systems. Since then BSPDEs appeared in a large amount of literature related to control theory as well as many other research fields. For example, in the study of stochastic maximum principle for

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stochastic PDEs or stochastic differential equations (SDEs) with partial information, the adjoint equations of Duncan–Mortensen–Zakai filtering equations are needed to solve first, which are actually BSPDEs. For this kind of discussions and applications, one can refer to [9,11,18,22,26], to name but a few. Moreover, by means of the classical duality argument, the controllability of stochastic evolution equations can be reduced to the observability estimate for BSPDEs, and this duality relation was utilized in e.g. [1,24]. Besides the applications in control theory, BSPDEs are also used to the stochastic process theory and mathematical finance, and we recommend the reader to see [5,8,14,15] for more details.

However, the solvability and the regularity of BSPDE, even for linear BSPDE, are tough problems due to the differential operators in the equation and its non-Markovian character. The recent work [7] by Du, Tang and Zhang made some progress and lifted the restrictions on the technical conditions for the Cauchy problem of linear degenerate BSPDEs. This work motivates us to consider the Cauchy problem of semi-linear degenerate BSPDEs under general settings. Actually, non-linear stochastic equations bear more application backgrounds without the exception of non-linear BSPDEs. For instance, Peng [21] discussed the Bellman dynamic principle for non-Markovian processes, whose corresponding backward stochastic Hamilton–Jacobi–Bellman equation is a fully non-linear BSPDE. Moreover, in many subjects of mathematical finance, such as imperfect hedging, portfolio choice, etc., non-linear BSPDEs appear as an important role and one can consult [16,17] for this aspect if interested.

Needless to say, more difficulties lay on the solvability of non-linear BSPDEs. In fact, the solvability of the solution to the fully non-linear BSPDE put forward in [21] is still an open problem, under general settings. Even for semi-linear BSPDE below we consider in this paper, only few work studied on it:

$$\begin{cases} du = -\big[\mathcal{L}u + \mathcal{M}q + f(t,x,u,q+u_x\sigma)\big]dt + q^k dW_t^k \\ u(T,x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where

$$\mathcal{L}u := a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu \quad \text{and} \quad \mathcal{M}q := \sigma^{ik}q^k_{x^i} + \nu^k q^k.$$

In 2002, Hu, Ma and Yong considered the semi-linear BSPDE of the above form, under some specific settings and technical conditions in [10]. For instance, they only considered one-dimensional equation and the coefficients σ, ν were independent of x. One of our goal in this paper is to lift these restrictions and derive the existence, uniqueness and regularity of semi-linear degenerate BSPDE without technical assumptions. Also we would like to indicate that the similar regularity of solutions are obtained in this paper, but much weaker regularity requirements on the coefficients are needed in comparison with [10]. Besides, Tang [23] is also concerned with semi-linear degenerate BSPDEs by the method of stochastic flows, but this method causes a cost of assuming differentiability of higher orders in x on the coefficients.

Our another motivation is to establish the correspondence between semi-linear degenerate BSPDE and FBSDE. It is well known that, in the Markovian framework, the Feynman–Kac formula for semi-linear equations was established by Peng [20] and Pardoux–Peng [19]. This Feynman–Kac formula demonstrates a correspondence between semi-linear PDE and FBSDE whose coefficients are all Markov processes. But in the non-Markovian framework, FBSDE does not correspond to a deterministic PDE any more, but a BSPDE instead, by stochastic calculus. Certainly, as an extension of the Feynman–Kac formula, this kind of correspondence is basically important, whether in Mathematical finance research field or in a potential application