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Small mass asymptotic for the motion with vanishing friction

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Abstract

We consider the small mass asymptotic (Smoluchowski–Kramers approximation) for the Langevin equation with a variable friction coefficient. The friction coefficient is assumed to be vanishing within certain region. We introduce a regularization for this problem and study the limiting motion for the 1-dimensional case and a multidimensional model problem. The limiting motion is a Markov process on a projected space. We specify the generator and the boundary condition of this limiting Markov process and prove the convergence.

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1. Introduction

The Langevin equation

$$\mu \ddot{\boldsymbol{q}}_t^\mu = \boldsymbol{b}(\boldsymbol{q}_t^\mu) - \lambda \dot{\boldsymbol{q}}_t^\mu + \sigma(\boldsymbol{q}_t^\mu)\dot{\boldsymbol{W}}_t, \qquad \boldsymbol{q}_0^\mu = \boldsymbol{q} \in \mathbb{R}^n, \qquad \dot{\boldsymbol{q}}_0^\mu = \boldsymbol{p} \in \mathbb{R}^n, \tag{1.1}$$

describes the motion of a particle of mass μ in a force field $\boldsymbol{b}(\boldsymbol{q})$, $\boldsymbol{q} \in \mathbb{R}^n$, subject to random fluctuations and to a friction proportional to the velocity. Here $\boldsymbol{W}_t$ is the standard Wiener process

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in $\mathbb{R}^n$, $\lambda > 0$ is the friction coefficient. The vector field $\boldsymbol{b}(\boldsymbol{q})$ and the matrix function $\sigma(\boldsymbol{q})$ are assumed to be continuously differentiable and bounded together with their first derivatives. The matrix $a(\boldsymbol{q}) = (a_{ij}(\boldsymbol{q})) = \sigma(\boldsymbol{q})\sigma^*(\boldsymbol{q})$ is assumed to be non-degenerate.

It is assumed usually that the friction coefficient λ is a positive constant. Under this assumption, one can prove that $\boldsymbol{q}_t^\mu$ converges in probability as $\mu \downarrow 0$ uniformly on each finite time interval $[0, T]$ to an n-dimensional diffusion process $\boldsymbol{q}_t$: for any $\kappa, T > 0$ and any $\boldsymbol{p}_0^\mu = \boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{q}_0^\mu = \boldsymbol{q} \in \mathbb{R}^n$ fixed,

$$\lim_{\mu\downarrow 0} \mathbf{P}\left(\max_{0\le t\le T} |\boldsymbol{q}_t^\mu - \boldsymbol{q}_t|_{\mathbb{R}^d} > \kappa\right) = 0.$$

Here $\boldsymbol{q}_t$ is the solution of equation

$$\dot{\boldsymbol{q}}_t = \frac{1}{\lambda}\boldsymbol{b}(\boldsymbol{q}_t) + \frac{1}{\lambda}\sigma(\boldsymbol{q}_t)\dot{\boldsymbol{W}}_t, \qquad \boldsymbol{q}_0 = \boldsymbol{q}_0^\mu = \boldsymbol{q} \in \mathbb{R}^n. \tag{1.2}$$

The stochastic term in (1.2) should be understood in the Itô sense.

The approximation of $\boldsymbol{q}_t^\mu$ by $\boldsymbol{q}_t$ for $0 < \mu \ll 1$ is called the Smoluchowski–Kramers approximation. This is the main justification for replacement of the second order equation (1.1) by the first order equation (1.2). The price for such a simplification, in particular, consists of certain non-universality of Eq. (1.2): the white noise in (1.1) is an idealization of a more regular stochastic process $\dot{\boldsymbol{W}}_t^\delta$ with correlation radius $\delta \ll 1$ converging to $\dot{\boldsymbol{W}}_t$ as $\delta \downarrow 0$. Let $\boldsymbol{q}_t^{\mu,\delta}$ be the solution of Eq. (1.1) with $\dot{\boldsymbol{W}}_t$ replaced by $\dot{\boldsymbol{W}}_t^\delta$. Then limit of $\boldsymbol{q}_t^{\mu,\delta}$ as $\mu, \delta \downarrow 0$ depends on the relation between μ and δ. Say, if first $\delta \downarrow 0$ and then $\mu \downarrow 0$, the stochastic integral in (1.2) should be understood in the Itô sense; if first $\mu \downarrow 0$ and then $\delta \downarrow 0$, $\boldsymbol{q}_t^{\mu,\delta}$ converges to the solution of (1.2) with stochastic integral in the Stratonovich sense (see, for instance, [5]).

We considered in [6] the case of a variable friction coefficient $\lambda = \lambda(\boldsymbol{q})$. We assumed in that work that $\lambda(\boldsymbol{q})$ is smooth and $0 < \lambda_0 \le \lambda(\boldsymbol{q}) \le \Lambda < \infty$. It turns out that in this case the solution $\boldsymbol{q}_t^\mu$ of (1.1) does not converge, in general, to the solution of (1.2) with $\lambda = \lambda(\boldsymbol{q})$, so that the Smoluchowski–Kramers approximation should be modified. In order to do this, we considered in [6] Eq. (1.1) with $\dot{\boldsymbol{W}}_t$ replaced by $\dot{\boldsymbol{W}}_t^\delta$ described above:

$$\mu\ddot{\boldsymbol{q}}_t^{\mu,\delta} = \boldsymbol{b}(\boldsymbol{q}_t^{\mu,\delta}) - \lambda(\boldsymbol{q}_t^{\mu,\delta})\dot{\boldsymbol{q}}_t^{\mu,\delta} + \sigma(\boldsymbol{q}_t^{\mu,\delta})\dot{\boldsymbol{W}}_t^\delta, \qquad \boldsymbol{q}_0^{\mu,\delta} = \boldsymbol{q}, \qquad \dot{\boldsymbol{q}}_0^{\mu,\delta} = \boldsymbol{p}. \tag{1.3}$$

It was proved in [6] that after such a regularization, the solution of (1.3) has a limit $\boldsymbol{q}_t^\delta$ as $\mu \downarrow 0$, and $\boldsymbol{q}_t^\delta$ is the unique solution of the equation obtained from (1.3) as $\mu = 0$:

$$\dot{\boldsymbol{q}}_t^\delta = \frac{1}{\lambda(\boldsymbol{q}_t^\delta)}\boldsymbol{b}(\boldsymbol{q}_t^\delta) + \frac{1}{\lambda(\boldsymbol{q}_t^\delta)}\sigma(\boldsymbol{q}_t^\delta)\dot{\boldsymbol{W}}_t^\delta, \qquad \boldsymbol{q}_0^\delta = \boldsymbol{q}. \tag{1.4}$$

Now we can take $\delta \downarrow 0$ in (1.4). As the result we get the equation

$$\dot{\boldsymbol{q}}_t = \frac{1}{\lambda(\boldsymbol{q}_t)}\boldsymbol{b}(\boldsymbol{q}_t) + \frac{1}{\lambda(\boldsymbol{q}_t)}\sigma(\boldsymbol{q}_t) \circ \dot{\boldsymbol{W}}_t, \qquad \boldsymbol{q}_0 = \boldsymbol{q}, \tag{1.5}$$

where the stochastic term should be understood in the Stratonovich sense. We have, for any $\delta, \kappa, T > 0$ fixed and any $\boldsymbol{p}_0^{\mu,\delta} = \boldsymbol{p}$ fixed, that

$$\lim_{\mu\downarrow 0} \mathbf{P}\left(\max_{0\le t\le T} |\boldsymbol{q}_t^{\mu,\delta} - \boldsymbol{q}_t^\delta|_{\mathbb{R}^d} > \kappa\right) = 0,$$