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Stochastic Processes and their Applications 123 (2013) 110–130
stochastic processes and their applications
www.elsevier.com/locate/spa

Hitting times for the perturbed reflecting random walk

Laurent Serlet*

Clermont Université, Université Blaise Pascal, Laboratoire de Mathématiques (CNRS umr 6620), Complexe scientifique des Cézeaux, BP 80026, 63171 Aubière, France

Received 19 October 2011; received in revised form 4 September 2012; accepted 4 September 2012
Available online 10 September 2012

Abstract

We consider a nearest neighbor random walk on $\mathbb{Z}$ which is reflecting at 0 and perturbed when it reaches its maximum. We compute the law of the hitting times and derive many corollaries, especially invariance principles with (rather) explicit descriptions of the asymptotic laws. We also obtain some results on the almost sure asymptotic behavior. As a by-product one can derive results on the reflecting Brownian motion perturbed at its maximum.

Keywords: Perturbed random walk; Once reinforced random walk; Perturbed Brownian motion; Hitting times; Invariance principle; Recurrence; Law of the iterated logarithm

1. Introduction and statement of the results

Processes with reinforcement have already generated an important amount of literature. Pemantle gives in [13] a very pleasant survey with lots of references. Reinforced random walks on a graph where introduced by Diaconis in 1987 with an edge reinforcement scheme. Other reinforcement schemes were introduced later, for instance sequence-type reinforcement as in [7]. Many questions remain open concerning reinforced random walks, especially in dimension greater than 1. In the present paper we stay in dimension 1 and we concentrate on the simplest case: the once reinforced random walk which is a random walk perturbed when reaching its extrema and more particularly its variant obtained by reflection at 0. This walk will be called a perturbed reflecting random walk (*PRRW*). Let us give a precise definition.

* Tel.: +33 0 473407093.
E-mail address: Laurent.Serlet@math.univ-bpclermont.fr.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2012.09.003

For any real valued process $(X_n)_{n\geq 0}$, we denote by $\mathcal{F}_n^X$ the σ-algebra generated by $X_0, X_1, \ldots, X_n$ and we set $\overline{X}_n = \max\{X_0, X_1, \ldots, X_n\}$. The PRRW with reinforcement parameter $r \in (-1, 1)$ is a process $(X_n)_{n\geq 0}$ taking its values on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that, for every $n \geq 0$, $X_{n+1} \in \{X_n - 1, X_n + 1\}$ and the transition probability $\mathbb{P}\left(X_{n+1} = X_n + 1 | \mathcal{F}_n^X\right)$ is equal to

- $1/2$ if $0 < X_n < \overline{X}_n$
- $(1-r)/2$ if $X_n = \overline{X}_n$ and $n \geq 1$
- 1 if $X_n = 0$

and moreover we suppose $X_0 = 0$. Of course the case $r = 0$ corresponds to the reflecting standard random walk (*RSRW*). We will also use the quantity $\beta = (1+r)/(1-r)$ to simplify some formulas. We interpret the case $r > 0$ as a self attractive walk called in the literature a reinforced random walk—whereas for $r < 0$ the walk is self repulsive and often called a negatively reinforced random walk. We summarize it in the array below.

r	$-1 \nearrow 0$	0	$0 \nearrow 1$
$\beta = (1+r)/(1-r)$	$0 \nearrow 1$	1	$1 \nearrow +\infty$
Terminology	negatively reinforced	standard	reinforced
Interpretation	self repellent		self attracting

(1)

Davis [8,9] has shown that a random walk perturbed when reaching its extrema converges, after the same rescaling as in Donsker's Theorem, toward a continuous time process called perturbed Brownian motion. This process has been studied by many authors, see for instance [12,3, 16,8,9,14,4,5] and the references therein. In our case where reflection at 0 is added, the continuous time limit is the solution of the equation

$$W_t = B_t + \alpha \sup_{s\leq t} W_s + \frac{1}{2} L_t^W \tag{2}$$

where $(L_t^W)_{t\geq 0}$ is the local time process at level 0.

The goal of the present paper is to study the PRRW via an excursion point of view. Since the PRRW behaves as a standard random walk when it is below the maximum, we concentrate on the study of the maximum process. This leads to the study of the hitting time process $(T_n)_{n\geq 0}$ defined, as usual, by $T_n = \inf\{k \geq 0;\ X_k = n\}$. Our starting point is an elementary representation of these hitting times using the excursions below the already visited levels (see Section 3). Most of the results in the paper are in fact derived from this representation and we will state them in the rest of this section.

We start with an invariance principle for the rescaled hitting time process.

For a process with trajectories in the space $\mathbb{D}([0, +\infty), \mathbb{R})$ of càdlàg functions, "convergence in law" means weak convergence of probability laws on this space endowed with the usual Skorohod topology.

Theorem 1. *Let $(\tau_t^n)_{t\geq 0}$ be the rescaled process of the hitting times of the PRRW defined by*

$$\tau_t^n = \frac{1}{n^2} T_{[nt]} \tag{3}$$

where $[\cdot]$ denotes the integer part.