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Large volatility-stabilized markets

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Abstract

We investigate the behavior of systems of interacting diffusion processes, known as volatility-stabilized market models in the mathematical finance literature, when the number of diffusions tends to infinity. We show that, after an appropriate rescaling of the time parameter, the empirical measure of the system converges to the solution of a degenerate parabolic partial differential equation. A stochastic representation of the latter in terms of one-dimensional distributions of a time-changed squared Bessel process allows us to give an explicit description of the limit.

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Keywords: Interacting diffusion processes; Hydrodynamic limit; Volatility-stabilized models; Bessel processes; Degenerate parabolic partial differential equations

1. Introduction

Recently, Fernholz and Karatzas [9] have introduced two kinds of systems of interacting diffusion processes, the volatility-stabilized market models and the rank-based market models, in the context of stochastic portfolio theory. Both of them serve as models for the evolution of capitalizations in equity markets and incorporate the fact that stocks of firms with smaller market capitalization tend to have higher rates of returns and be more volatile. In a previous paper [25] the author gave a description of the joint dynamics of the market capitalizations in rank-based models, when the number of firms tends to infinity (see also [13] for related results). Here, the corresponding limit is investigated in the context of volatility-stabilized models. For an analysis of arbitrage opportunities in these models we refer the reader to [8,2].

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The dynamics of the capitalizations in volatility-stabilized models is given by the unique weak solution to the following system of stochastic differential equations:

$$dX_i(t) = \frac{\eta}{2} S(t)dt + \sqrt{X_i(t)S(t)}\, dW_i(t), \quad 1 \le i \le N, \tag{1}$$

which is endowed with an initial distribution of the vector $(X_1(0), \dots, X_N(0))$ on $[0, \infty)^N$. Hereby, η is a real number greater than 1, $S(t) = X_1(t) + \cdots + X_N(t)$ and $W_1, \dots, W_N$ is a collection of N independent standard Brownian motions. We refer the reader to Section 12 of [9] for a construction of a weak solution to (1) and an explanation of why it is unique.

We will analyze the limit of the path of empirical measures $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}$ corresponding to (1), after a suitable rescaling of the time parameter, when N tends to infinity. The slowdown of the time by a factor of N is needed to observe a non-degenerate limiting behavior. Heuristically, this can be inferred from the appearance of the process $S(t) = X_1(t) + \cdots + X_N(t)$, an order N object, in the drift and diffusion coefficients of the processes $X_1, \dots, X_N$.

We show that the limit of the sequence of laws of $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t/N)}$, $N \in \mathbb{N}$, exists and that the limiting measure is supported on generalized solutions of the degenerate linear parabolic equation (5) below. Overcoming the problem of degeneracy, we show that the generalized solution of the Eq. (5) is unique. Having shown uniqueness, we use a stochastic representation of the solution of (5) to determine the latter explicitly.

Due to our results one may approximate the evolution of the capitalizations in a large volatility-stabilized market by the solution of the limiting Eq. (5). Moreover, in the context of stochastic portfolio theory (see e.g. [7,9]) one is interested in the behavior of the rank statistics of the vector $(X_1(t), \dots, X_N(t))$ of capitalizations. Since these are given by the $\frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}$-quantiles of the empirical measure $\frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}$, our results can also be used to approximate sample paths of any finite number of ranked capitalizations (or market weights) by the sample paths of the corresponding quantiles (or appropriate functions of those) of the solution to the partial differential equation (5). This complements the exact formulas for the transition probabilities of the market weights in volatility-stabilized markets given in [20], which allow us to simulate the vector of market weights at finitely many different points in time. In addition, the stochastic representation mentioned above shows that the solution to Eq. (5) is given by the one-dimensional distributions of a time-changed squared Bessel process and, thus, establishes a new connection between volatility-stabilized market models and squared Bessel processes (see [9] for further connections). The latter were analyzed in much detail in the works [22–24] among others.

Independently from the field of stochastic portfolio theory, systems of interacting diffusion processes play a major role in statistical physics. In particular, systems of diffusions interacting through their empirical measure (mean field) have been studied in the literature by many authors; see e.g. [11,15,4,10,14,16–18]. We remark that the system (1) can be cast into the framework of [11], since the drift and the diffusion coefficients in the ith equation of the system (1) can be expressed as functions of the empirical measure of the particle system and the position of the ith particle. However, the generator of the particle system is not uniformly elliptic on $[0, \infty)^N$ and the same is true on $[0, \infty)$ for the elliptic differential operator on the right-hand side of Eq. (5). For this reason, the results of [11] do not carry over directly to our setting. Nonetheless, we adapt some of the techniques developed there to our case.

The time-varying mass partition

$$\alpha_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_N(t)}, \quad 1 \le i \le N \tag{2}$$