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# Extremes of the standardized Gaussian noise

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**Abstract**

Let $\{\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}^d\}$ be a $d$-dimensional array of independent standard Gaussian random variables. For a finite set $A \subset \mathbb{Z}^d$ define $\mathbb{S}(A) = \sum_{\boldsymbol{k} \in A} \xi_{\boldsymbol{k}}$. Let $|A|$ be the number of elements in $A$. We prove that the appropriately normalized maximum of $\mathbb{S}(A)/\sqrt{|A|}$, where $A$ ranges over all discrete cubes or rectangles contained in $\{1, \dots, n\}^d$, converges in law to the Gumbel extreme-value distribution as $n \to \infty$. We also prove a continuous-time counterpart of this result.


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## 1. Introduction and statement of results

Let $\{\xi_k, k \in \mathbb{N}\}$ be independent standard Gaussian random variables. Denote by $S_k = \xi_1 + \dots + \xi_k$ the corresponding random walk and let

$$L_n = \max_{0 \le i < j \le n} \frac{S_j - S_i}{\sqrt{j - i}}. \tag{1}$$

It has been shown by Siegmund and Venkatraman [24] that for every $\tau \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{P}\left[ L_n \le \sqrt{2 \log n} + \frac{\frac{1}{2} \log \log n + \log \frac{H}{2\sqrt{\pi}} + \tau}{\sqrt{2 \log n}} \right] = \mathrm{e}^{-\mathrm{e}^{-\tau}}, \tag{2}$$

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where $H \in (0, \infty)$ is some constant. A different proof of the same result has been given in [11] where also the following continuous-time counterpart of (2) can be found. Let $\{B(t), t \ge 0\}$ be a standard Brownian motion. For $n > 1$ define

$$M_n = \sup_{\substack{x,y\in[0,n]\\ y-x\ge 1}} \frac{B(y)-B(x)}{\sqrt{y-x}}. \tag{3}$$

Then, for every $\tau \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[M_n \le \sqrt{2\log n} + \frac{\frac{3}{2}\log\log n - \log(2\sqrt{\pi}) + \tau}{\sqrt{2\log n}}\right] = \mathrm{e}^{-\mathrm{e}^{-\tau}}. \tag{4}$$

Almost sure laws of large numbers for $L_n$, $M_n$ and related quantities have been obtained in [23,26,12,13]. An analogue of $L_n$ in the case of heavy-tailed random variables has been studied in [16].

Our aim here is to prove multidimensional counterparts of (2) and (4). We shall be interested in the maximum of discrete-time or continuous-time $d$-dimensional Gaussian noise standardized by the square root of its variance. The maximum is taken over some family of $d$-dimensional subsets. Here, we shall consider two families of subsets, rectangles and cubes, in discrete and continuous settings. Both families are multidimensional generalizations of the collection of one-dimensional intervals.

Let us state our discrete-time results first. Let $\{\xi_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}^d\}$ be a $d$-dimensional array of independent standard Gaussian random variables. Given a finite set $A \subset \mathbb{Z}^d$ we define

$$\mathbb{S}(A) = \sum_{\boldsymbol{k}\in A} \xi_{\boldsymbol{k}}. \tag{5}$$

Note that $\operatorname{Var}\mathbb{S}(A) = |A|$, where $|A|$ is the number of elements in the set $A$.

A set of the form $\{x_1, \dots, x_1 + h\} \times \dots \times \{x_d, \dots, x_d + h\}$, where $x_1, \dots, x_d \in \mathbb{Z}$ and $h \in \mathbb{N} \cup \{0\}$, is called a $d$-dimensional discrete cube. Denote by $\mathfrak{C}^d$ the set of all discrete $d$-dimensional cubes and let $\mathfrak{C}_n^d$ be the set of all discrete $d$-dimensional cubes contained in $\{1, \dots, n\}^d$, where $n \in \mathbb{N}$. Define

$$u_n(\tau) = \sqrt{2d\log n} + \frac{\frac{1}{2}\log(d\log n) + \log\frac{(2d)^d J_d}{\sqrt{\pi}} + \tau}{\sqrt{2d\log n}}, \quad \tau \in \mathbb{R}, \tag{6}$$

where $J_d \in (0, \infty)$ is a constant defined in Lemma 4.1 below.

**Theorem 1.1.** *For every $\tau \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left[\max_{A\in\mathfrak{C}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \le u_n(\tau)\right] = \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

A set of the form $\{x_1, \dots, y_1\} \times \dots \times \{x_d, \dots, y_d\}$, where $x_i, y_i \in \mathbb{Z}$ and $x_i \le y_i$ for all $1 \le i \le d$, is called a $d$-dimensional discrete rectangle. Note that a discrete cube is a discrete rectangle whose sides have equal lengths. Denote by $\mathfrak{R}^d$ the collection of all discrete $d$-dimensional rectangles and let $\mathfrak{R}_n^d$ be the set of all discrete $d$-dimensional rectangles contained