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ScienceDirect
Stochastic Processes and their Applications 121 (2011) 609–629
stochastic processes and their applications
www.elsevier.com/locate/spa

The contact process on the complete graph with random vertex-dependent infection rates

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Received 5 May 2010; received in revised form 21 October 2010; accepted 8 November 2010
Available online 16 November 2010

Abstract

We study the contact process on the complete graph on n vertices where the rate at which the infection travels along the edge connecting vertices i and j is equal to $\lambda w_i w_j / n$ for some $\lambda > 0$, where w_i are i.i.d. vertex weights. We show that when $E[w_1^2] < \infty$ there is a phase transition at $\lambda_c > 0$ such that for $\lambda < \lambda_c$ the contact process dies out in logarithmic time, and for $\lambda > \lambda_c$ the contact process lives for an exponential amount of time. Moreover, we give a formula for λ_c and when $\lambda > \lambda_c$ we are able to give precise approximations for the probability that a given vertex is infected in the quasi-stationary distribution.

Our results are consistent with a non-rigorous mean field analysis of the model. This is in contrast to some recent results for the contact process on power law random graphs where the mean field calculations suggested that $\lambda_c > 0$ when in fact $\lambda_c = 0$.

MSC: primary 60K35; secondary 60K37; 82B26; 05C80

Keywords: Contact process; Random environment; Phase transition

1. Introduction

The contact process is a simple model for the spread of a disease. The standard model of the contact process on a graph $G = (V, E)$ is described informally as follows. Fix a parameter $\lambda > 0$ and a set of vertices $A \subset V$. At time $t = 0$ only the vertices in A are infected. As time

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.11.003

progresses, each uninfected vertex x becomes infected at rate equal to λ times the number of currently infected neighbors, and each infected vertex becomes a healthy (uninfected) vertex at rate 1. More formally, the contact process is a continuous time Markov process $\eta_t \in \{0, 1\}^V$ with generator

$$\mathcal{L} f(\eta) = \sum_{x\in V} \left(\eta(x) + (1 - \eta(x))\lambda \sum_{y\sim x} \eta(y) \right) \left(f(\eta^x) - f(\eta) \right),$$

where $f : \{0, 1\}^V \to \mathbb{R}$ is a bounded function, $x \sim y$ means that vertices x and y are connected by an edge in E, and η^x is the configuration obtained from η by switching the value of $\eta(x)$. That is,

$$\eta^x(z) = \begin{cases} \eta(z) & z \neq x \\ 1 - \eta(z) & z = x. \end{cases} \tag{1}$$

The contact process is also sometimes referred to as the susceptible–infected–susceptible (SIS) epidemic model.

The behavior of the contact process depends on the parameter λ, and as λ increases the infection spreads faster and it takes a longer amount of time for the contact process to die out (i.e., reach the absorbing state of all healthy vertices). It is then natural to ask whether there is a critical value of λ for which the contact process exhibits a phase transition. The contact process on the integer lattice $\mathbb{Z}^d$ has been well studied, and it is known that there is a $\lambda_c > 0$ such that the contact process started with a single vertex infected dies out with probability 1 if $\lambda < \lambda_c$ and survives forever with positive probability if $\lambda > \lambda_c$ [5]. On any finite graph the contact process always eventually dies out, and thus it is not immediately clear how to define a phase transition. However, for the contact process on $[-n, n]^d \subset \mathbb{Z}^d$ it is known that for $\lambda < \lambda_c$ (where λ_c is the critical parameter for the contact process on $\mathbb{Z}^d$) the contact process dies out by time $C \log n$ with high probability, whereas for $\lambda > \lambda_c$ the contact process survives for time $\exp\{cn^d\}$ with positive probability [5]. In general, one says that the contact process on a family of finite graphs is sub-critical if the time until the infection dies out is logarithmic in the number of vertices and is super-critical if with positive probability the infection survives for a time that is larger than any polynomial in the number of vertices of the graph. The critical value λ_c then identifies the phase transition of the contact process from sub-critical to super-critical.

The contact process has also been studied on graphs other than $\mathbb{Z}^d$. Probably the first such work was done by Pemantle on the contact process on infinite trees [9]. The contact process has also been studied on certain non-homogeneous classes of graphs. Recently, Chatterjee and Durrett [3] and Berger et al. [2] considered the contact process on two different models of power law random graphs. A power law random graph is a general term denoting a class of graphs where the distribution of the degree of a typical vertex has tails that decay like $Ck^{-\alpha-1}$ as $k \to \infty$ for some $\alpha > 1$ and $C > 0$. Physicists had previously studied the contact process on power law random graphs using non-rigorous mean field calculations and concluded that if $\alpha > 3$ then there was a critical value $\lambda_c > 0$ identifying a phase transition [8,7]. However, for the two kinds of power law random graphs studied in [2,3] it was shown that in fact $\lambda_c = 0$ (i.e., the contact process survives for a long time for any $\lambda > 0$). The long time survival of the contact process implies the existence of a quasi-stationary distribution. The mean field calculations suggest that the average density $\rho(\lambda)$ of infected sites in the quasi-stationary distribution satisfies $\rho(\lambda) \sim C\lambda^\beta$ for some $\beta > 0$ as $\lambda \to \lambda_c^+$. However, upper and lower bounds on $\rho(\lambda)$ calculated in [3] show that the "critical exponent" β must be different from the mean field predictions.