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Functions of bounded variation on the classical Wiener space and an extended Ocone–Karatzas formula

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Abstract

We prove an extension of the Ocone–Karatzas integral representation, valid for all BV functions on the classical Wiener space. We also establish an elementary chain rule formula and combine the two results to compute explicit integral representations for some classes of BV composite random variables.

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Keywords: BV functions; Wiener space; Ocone–Karatzas formula

1. Introduction

Functions of bounded variation (BV) in a Gaussian Banach space setting were first investigated by Fukushima and Hino in [6,7], using techniques from the theory of Dirichlet forms. More recently, Ambrosio and his co-workers have given an alternative approach, in [3], by adapting techniques from geometric measure theory.

The most important example of an infinite-dimensional Gaussian space is given by the classical Wiener space $(\Omega, \mathcal{A}, \mathbb{P})$, i.e. the space of trajectories of the Wiener process. This was the setting where the Malliavin calculus was originally developed and still most of its applications are formulated. Since BV functions generalize Malliavin differentiable functions, we specialize here the general results valid for all Gaussian spaces, work on explicit examples and consider new problems which appear naturally, in connection with stochastic analysis. Thus, all the results

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given here are formulated in this setting, though some of them are certainly valid in any abstract Wiener space.

One of the aims of this paper is to study how the theory of BV functions can be a useful additional tool, even when dealing with classical problems. We believe that the extension of the Ocone–Karatzas formula, Theorem 16, can be regarded as the best example, in this direction, among those presented in this paper.

Roughly speaking, a real function f defined on the classical Wiener space is BV if it admits an $L^2(0, T)$-valued measure Df which plays the role of a Malliavin derivative, so that an integration by parts identity holds true. Generalizing the situation of differentiable functions, where the Malliavin derivative is a stochastic process, Df can be seen as a measure on the product space $(\Omega \times [0, T], \mathcal{A} \otimes \mathcal{B}(0, T))$ and processes can be integrated with respect to it.

Notably, Df can be not absolutely continuous with respect to $\mathbb{P} \otimes \lambda$ (where λ is the Lebesgue measure). However, if we introduce the strictly predictable σ-algebra $\mathcal{P}$, which is slightly smaller than the usual σ-algebra of predictable sets in $\Omega \times [0, T]$, then Df restricted to $\mathcal{P}$ becomes absolutely continuous with respect to $\mathbb{P} \otimes \lambda$. Moreover, if $H = (H_s)_{0 \le s \le T}$ is a version of the density, then

$$f = \mathbb{E}[f] + \int_0^T H_s dW_s.$$

This is, informally, the content of Theorem 16, that is our extension of the classical Ocone–Karatzas formula, originally proved in [8], which identifies the integrand in the Itô representation of a random variable in terms of its Malliavin derivative. In the differentiable case, one usually writes $H_s = \mathbb{E}(\partial_s f | \mathcal{F}_s)$, for λ-a.e. $s \in [0, T]$. Proposition 13 shows that an similar result holds true in the BV case, which can be useful when dealing with concrete cases, although from a higher point of view the process H should be considered as the (dual) predictable projection of the measure Df.

BV functions can be helpful when dealing with a composite function $f = \phi \circ g$, where g is a real random variable, differentiable in the Malliavin sense, and ϕ is a Euclidean BV function. In such a case, as in the classical theory of Euclidean BV functions, many problems for general functions can be reduced to the case of indicator functions of level sets $\{x > g\}$, which can be shown to be BV, under certain assumptions. Moreover, an explicit chain rule formula can be obtained (Propositions 8 and 10). We remark here that this chain rule is still very distant from the deep results which can be obtained in the Euclidean setting, but it can be useful when dealing with applications.

Indeed, as an application, we combine the extended Ocone–Karatzas formula and the chain rule, to obtain explicit representations for some functionals of the Wiener process (Propositions 20 and 22). While the former result is well-known, nevertheless we believe that the theory of BV functions provides, we believe, a clear proof without advanced technical results, such as the theory of distributions on Wiener spaces.

This paper is organized as follows: in Section 2, we fix some notation and provide the definition of BV functions together with the main approximation result, Theorem 6, without proof. Other technical results, e.g. on the Orlicz space $L \log^{1/2} L$, are collected, for the convenience of the reader. In Section 3, we investigate a chain rule for a special class of BV functions. Here, we use an approximation result for Euclidean BV functions, though an elementary proof is given for the special case of indicator functions of level sets. In Section 4, the extended Ocone–Karatzas formula is established, after some remarks on the notion of predictability. In Section 5, applications and examples are discussed.