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A local limit theorem for a transient chaotic walk in a frozen environment☆

Lasse Leskelä[a,*], Mikko Stenlund[b,c]

[a] *Department of Mathematics and Systems Analysis, Aalto University, PO Box 11100, 00076 Aalto, Finland*
[b] *Courant Institute of Mathematical Sciences, New York, NY 10012, USA*
[c] *Department of Mathematics and Statistics, PO Box 68, 00014 University of Helsinki, Finland*

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Abstract

This paper studies particle propagation in a one-dimensional inhomogeneous medium where the laws of motion are generated by chaotic and deterministic local maps. Assuming that the particle's initial location is random and uniformly distributed, this dynamical system can be reduced to a random walk in a one-dimensional inhomogeneous environment with a forbidden direction. Our main result is a local limit theorem which explains in detail why, in the long run, the random walk's probability mass function does not converge to a Gaussian density, although the corresponding limiting distribution over a coarser diffusive space scale is Gaussian.

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* Corresponding author. Tel.: +358 9 4702 3079; fax: +358 9 4702 3016.
E-mail addresses: lasse.leskela@iki.fi (L. Leskelä), mikko@cims.nyu.edu (M. Stenlund).
URLs: http://www.iki.fi/lsl/ (L. Leskelä), http://www.math.helsinki.fi/mathphys/mikko.html (M. Stenlund).

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1. Introduction

1.1. A chaotic dynamical system

This paper studies a particle moving in a continuous inhomogeneous medium which is composed of a linear chain of cells modeled by the unit intervals $[k, k+1)$ of the positive real line. Each interval $[k, k+1)$ is assigned a label ω_k and a map U_{ω_k} which determines the dynamics of the particle as long as the particle remains in the interval. The sequence of labels $\omega = (\omega_k)_{k\in\mathbb{Z}_+}$, called an *environment*, is assumed to be either nonrandom, or a realization of a random sequence that is frozen during the particle's lifetime.

We are interested in the case in which the local dynamical rules U_{ω_k} are chaotic in the sense that the distance between two initially nearby particles grows at an exponential rate. More concretely, we shall focus on a model where a particle located at $x_n \in [k, k+1)$ at time n jumps to $x_{n+1} = k + U_{\omega_k}(x_n - k)$. Here $\omega_k \in (0, 1)$ and U_{ω_k} is the piecewise affine map from $[0, 1)$ onto $[0, 2)$ such that $U_{\omega_k}[0, 1-\omega_k) = [0, 1)$ and $U_{\omega_k}[1-\omega_k, 1) = [1, 2)$. The dynamical system generated by the local rules is compactly expressed by $x_{n+1} = \mathcal{U}_\omega(x_n)$, where the global map $\mathcal{U}_\omega$ on the positive real line is defined by

$$\mathcal{U}_\omega(x) = [x] + U_{\omega_{[x]}}(x - [x]), \tag{1.1}$$

and $[x]$ denotes the integral part of x; see Fig. 1.

The above model belongs to the realm of extended dynamical systems, a somewhat vaguely defined yet highly active field of research (e.g. Chazottes and Fernandez [9]). Telltale characteristics of such systems are a noncompact or high-dimensional phase space and the lack of relevant finite invariant measures. Our principal motivation is to study the impact of environment inhomogeneities on the long-term behavior of extended dynamical systems. Concrete models include neural oscillator networks (Lin, Shea-Brown, and Young [27]) and the Lorentz gas with randomly placed scatterers (Chernov and Dolgopyat [11]; Cristadoro, Lenci, and Seri [12] to name a few). In this paper, we shall restrict the analysis to the affine dynamical model in (1.1), to keep the presentation simple and clear.

1.2. Random initial data

Because the local maps U_{ω_k} are chaotic, predicting the particle's future location with any useful accuracy over any reasonably long time horizon would require a precise knowledge of its initial position—a sheer impossibility in practice. Therefore, it is natural to take the statistical point of view and study the stochastic process defined by

$$\begin{aligned} x_0 &\stackrel{d}{=} \text{Uniform}[0, 1), \\ x_{n+1} &= \mathcal{U}_\omega(x_n). \end{aligned} \tag{1.2}$$

To analyze the time evolution of the above process, we must impose some regularity conditions on the environment. In particular, those conditions guarantee ballistic motion, and one might guess that the distribution approaches Gaussian in the long run. To test this hypothesis, we have plotted in Fig. 2 numerically computed histograms of x_n at time $n = 2^{13}$ in two frozen environments, using the intervals $[k, k+1)$ as bins. Rather surprisingly, the histograms do not appear Gaussian. A similar phenomenon was recently observed by Simula and Stenlund [33,34].