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On the long time behavior of the TCP window size process

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Abstract

The TCP window size process appears in the modeling of the famous transmission control protocol used for data transmission over the Internet. This continuous time Markov process takes its values in $[0, \infty)$, and is ergodic and irreversible. It belongs to the additive increase–multiplicative decrease class of processes. The sample paths are piecewise linear deterministic and the whole randomness of the dynamics comes from the jump mechanism. Several aspects of this process have already been investigated in the literature. In the present paper, we mainly get quantitative estimates for the convergence to equilibrium, in terms of the W_1 Wasserstein coupling distance, for the process and also for its embedded chain.

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1. Introduction

The TCP protocol is one of the main data transmission protocols of the Internet. It has been designed to adapt to the various traffic conditions of the actual network. For a single-channel system, the maximum number of packets that can be sent without receiving an acknowledgement from the destination is given by a variable W, called the *congestion window size*. If all the W packets are successfully transmitted, then W is increased by 1; otherwise it is multiplied by $\delta \in [0, 1)$ (detection of a congestion). As shown in [6,11,21], a correct scaling of this process leads to a continuous time Markov process $X = (X_t)_{t \geqslant 0}$, called the general TCP window size process. This process has $[0, \infty)$ as state space and its infinitesimal generator is given, for any smooth function $f : [0, \infty) \to \mathbb{R}$, by

$$L(f)(x) = f'(x) + x \int_0^1 (f(hx) - f(x))H(\mathrm{d}h) \tag{1}$$

for some probability measure H supported in $[0, 1)$. This window size $(X_t)_t$ increases linearly (this is the f' part of L) until the reception of a congestion signal occurs, which forces the reduction of the window size by a multiplicative factor of law H or equal to δ in the simplest case (this is the jump part of L). The sample paths of X are deterministic between jumps, the jumps are multiplicative, and the whole randomness of the dynamics relies on the jump mechanism. Of course, the randomness of X may also come from a random initial value. The process $(X_t)_{t \geqslant 0}$ appears as an Additive Increase–Multiplicative Decrease process (AIMD), but also as a very special Piecewise Deterministic Markov Process (PDMP) initially introduced in [5]. In this direction, [17] gives a generalization of the scaling procedure to interpret various PDMPs as the limit of discrete time Markov chains and in [15] more general increase and decrease profiles are considered as models for TCP. In the real world (Internet), the AIMD mechanism allows a good compromise between the minimization of network congestion time and the maximization of mean throughput. See also [3] for a simplified TCP windows size model.

Our aim in this paper is to get quantitative estimates for the convergence to equilibrium of this general TCP window size process. This process X is ergodic and admits a unique invariant law, as can be checked using a suitable Lyapunov function (for instance $V(x) = 1 + x$; see e.g. [2,4,19] for the Meyn–Tweedie–Foster–Lyapunov technique and [12] for its applications to derive qualitative convergence to equilibrium for AIMDs). Nevertheless, this process is irreversible since time reversed sample paths are not sample paths and it has infinite support. This makes Meyn–Tweedie–Foster–Lyapunov techniques inefficient for the derivation of quantitative exponential ergodicity.

The *embedded chain* $\hat{X}$ of the process X is the sequence of its positions just after a jump. It is a homogeneous discrete time Markov chain with state space $[0, \infty)$. As already observed in [6], it is also the square root of a first-order auto-regressive process with non-Gaussian innovations and random coefficients. We obtain the following results concerning $\hat{X}$. We show first that it admits a unique invariant probability measure ν, and that it converges in law to ν given any (random) initial value $\hat{X}_0$. More precisely, using a coupling technique on trajectories, we prove an ergodic theorem of geometric convergence to equilibrium with respect to any Wasserstein distance. Then we provide non-asymptotic concentration bounds, thanks to Gross's logarithmic Sobolev inequalities.

Similarly, the continuous time process X admits a unique invariant probability measure μ, and converges in law to μ, for any (random) initial value X_0. The reader may find explicit series for the moments of μ and ν in [11,17,18]. Nevertheless, quantitative rates of convergence to