

Markov chain mixing time on cycles

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Received 14 December 2010; received in revised form 14 June 2011; accepted 20 July 2011

Available online 29 July 2011

Abstract

Mixing time quantifies the convergence speed of a Markov chain to the stationary distribution. It is an important quantity related to the performance of MCMC sampling. It is known that the mixing time of a reversible chain can be significantly improved by lifting, resulting in an irreversible chain, while changing the topology of the chain. We supplement this result by showing that if the connectivity graph of a Markov chain is a cycle, then there is an $\Omega(n^2)$ lower bound for the mixing time. This is the same order of magnitude that is known for reversible chains on the cycle.

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Keywords: Markov chain; Mixing time; Cycle; Non-reversible

1. Introduction, result formulation

The goal of this paper is to prove a lower bound on the mixing time of a family of Markov chains.

Mixing time is an important quantity directly related to the performance of numerous algorithms. In Markov chain Monte Carlo simulations (see [8]), mixing time can be interpreted as the time needed to generate a sample.

It turns out that running a local averaging algorithm is the same as following the evolution of the distribution of a certain Markov chain (see [10]). Again, the time needed to get within a certain neighborhood of a common value is quantified by the mixing time.

Motivated by these applications, the estimation of mixing time is in the center of interest.

Usually reversible Markov chains are used to solve these problems. It turns out that often a non-reversible variant can mix much faster. We go a step further on understanding the difference.

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Our result is expressed in a single theorem. We work with finite state discrete time Markov chains. We restrict the connectivity graph to a cycle, and allow arbitrary non-reversible transition probabilities such that the uniform distribution is invariant. Then there is a lower bound on the mixing time which has the same order of magnitude as the best lower bound for reversible chains.

We first formulate our result, then we show how it fits into existing literature.

Let us define the quantities and notions we use: If we start the chain with an initial distribution σ , let $\sigma^{(k)}$ denote the distribution after k steps. For the set of probability distributions on a finite base set Ω we use the notation $\mathcal{P}(\Omega)$.

Definition 1. Given two probability distributions μ and σ on Ω , the total variation distance is

$$\|\mu - \sigma\|_{\text{TV}} = \max_{A \subseteq \Omega} |\mu(A) - \sigma(A)|.$$

Definition 2. For a Markov chain with stationary distribution π and transition matrix $P = (P_{ij})$, with P_{ij} denoting the probability of moving from state i to state j , we define the mixing time of the chain as

$$t_{\text{mix}}(P, \varepsilon) = \max_{\sigma \in \mathcal{P}(\Omega)} \min \left\{ k : \|\sigma^{(k)} - \pi\|_{\text{TV}} \leq \varepsilon \right\}.$$

Note that this might be infinite if the Markov chain is non-ergodic.

We consider only the case when the stationary distribution is uniform. For the transition matrix this translates to the condition of being doubly stochastic.

A Markov chain is reversible if starting from the stationary distribution π , the probability of the consecutive pair (i, j) is the same as the probability of the consecutive pair (j, i) . Formally:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i, j.$$

The connectivity graph of a Markov chain is the graph formed by the states of the Markov chain as nodes and by undirected edges between i and j with $i \neq j$, if either of the transition probabilities p_{ij} or p_{ji} is nonzero. We do not include loops even if $p_{ii} > 0$. We shall also refer to this graph loosely as the topology of the Markov chain. In our case we assume this graph to be a subgraph of a single cycle.

For convenience, let us number the nodes according to the ordering on the cycle. We will interpret these numbers *mod n*.

We are now ready to state our result:

Theorem 1. Consider a Markov chain on a cycle with n nodes having a doubly stochastic transition matrix P . Then, with some global constant $C > 0$ we have

$$t_{\text{mix}}(P, 1/8) \geq Cn^2.$$

Note that our theorem covers all Markov chains, even non-reversible ones. Quantifiable bounds for the mixing time are often less sharp and/or harder to compute for non-reversible chains. Consider the classic method using the spectral gap $\gamma = (1 - \max_i(|\lambda_i|))$. For lazy reversible chains the mixing time turns out to be roughly $(1/\gamma) \log \varepsilon$, up to a factor of $-\log \min_i \pi_i$. For non-reversible chains, it may happen that the upper bound on TV distance does not converge to 0, so it does not give an upper bound on mixing time at all (see [9]).

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