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Almost sure asymptotics for the local time of a diffusion in Brownian environment

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Abstract

Here, we study the asymptotic behavior of the maximum local time $\mathbf{L}^*(t)$ of the diffusion in Brownian environment. Shi (1998) [17] proved that, surprisingly, the maximum speed of $\mathbf{L}^*(t)$ is at least $t\log(\log(\log t))$; whereas in the discrete case, it is t. We show that $t\log(\log(\log t))$ is the proper rate and that for the minimum speed the rate is the same as in the discrete case (see Dembo et al. (2007) [6]) namely $t/\log(\log(\log t))$. We also prove a localization result: almost surely for large time, the diffusion has spent almost all the time in the neighborhood of four points which only depend on the environment.

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1. Introduction

Let $(W(x), x \in \mathbf{R})$ be a two-sided one-dimensional Brownian motion on $\mathbf{R}$ with $W(0) = 0$. A *diffusion process in the environment* W is a process $(X(t), t \in \mathbf{R}^+)$ whose infinitesimal generator given W is

$$\frac{1}{2}e^{W(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-W(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

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Note that, if W were differentiable, $(\mathbf{X}(t), t \in \mathbf{R}^+)$ would be the solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}\mathbf{X}(t) = \mathrm{d}\beta(t) - \dfrac{1}{2}W'(\mathbf{X}(t))\mathrm{d}t, \\ \mathbf{X}(0) = 0 \end{cases}$$

in which β is a standard one-dimensional Brownian motion independent of W. Of course as we choose for W a Brownian motion, the previous equation does not have any rigorous sense, but it explains the denomination *environment* for W.

This process was first introduced by Schumacher [16] and Brox [4]. It is recurrent and sub-diffusive with the asymptotic behavior in $(\log t)^2$. Moreover, Brox showed in [4] that $\mathbf{X}$ has the property to be localized in the neighborhood of a point $m_{\log t}$ only depending on t and W. Later, Tanaka [22,21] and Hu [9] obtained deeper localization results. The limit law of $m_{\log t}/(\log t)^2$ and therefore of $\mathbf{X}(t)/(\log t)^2$ is independently made explicit by Kesten [12] and Golosov [8].

Kesten and Golosov's aim was actually to determine the limit law of a random walk in random environment, introduced by Solomon [19], which is often considered as the discrete analogue of Brox's model. Sinai [18] proved that this random walk $(S_n, n \in \mathbf{N})$, now called Sinai's walk, has the same limit distribution as Brox's. Hu and Shi [10] got the almost sure rates of convergence of the lim sup and lim inf of $\mathbf{X}$ and S. It appears that these rates are the same.

In the present paper, we are interested in the local time of the diffusion $\mathbf{X}$. This process, denoted as $(\mathbf{L}(t, x), t \geq 0, x \in \mathbf{R})$, is the density of the occupation measure of $\mathbf{X}$: $\mathbf{L}$ is the unique a.s. jointly continuous process such that for each Borel set A and for any $t \geq 0$,

$$\nu_t(A) := \int_0^t \mathbf{1}_A(\mathbf{X}_s)\mathrm{d}s = \int_{\mathbf{R}} \mathbf{1}_A(x)\mathbf{L}(t, x)\mathrm{d}x. \tag{1}$$

We shall see later that such a process exists. The first results on the behavior of $\mathbf{L}$ can be found in [17] and [11]. In particular, Hu and Shi proved in [11] that for any $x \in \mathbf{R}$,

$$\frac{\log(\mathbf{L}(t, x))}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U}, t \to +\infty$$

where U and $\hat{U}$ are independent variables uniformly distributed in $(0, 1)$ and $\xrightarrow{\mathcal{L}}$ is the convergence in law. Note that in the same paper it is also proved that Sinai's walk local time ξ has the same behavior. The process ξ is the time spent by S at x before time n for $n \in \mathbf{N}$ and $x \in \mathbf{Z}$: $\xi(n, x) := \sum_{k=0}^{n} \mathbf{1}_{S_k = x}$. This result shows that the local time in a fixed point can vary a lot. So to study localization, a good quantity to look at is the maximum of the local time of $\mathbf{X}$,

$$\mathbf{L}^*(t) := \sup_{x \in \mathbf{R}} \mathbf{L}(t, x).$$

Shi was the first one to be interested in this process; in [17], he gave a lower bound on the lim sup behavior. Almost surely,

$$\limsup_{t \to \infty} \frac{\mathbf{L}^*(t)}{t \log_3(t)} \geq \frac{1}{32}$$

where, for any $i \in \mathbf{N}^*$, $\log_{i+1} = \log \circ \log_i$ and $\log_1 = \log$. In the same paper, he computed the similar rate in the discrete case: define for $n \in \mathbf{N}$, $\xi^*(n) := \sup_{x \in \mathbf{Z}} \xi(n, x)$, there is the constant