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Strong approximation of partial sums under dependence conditions with application to dynamical systems

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Abstract

In this paper, we obtain precise rates of convergence in the strong invariance principle for stationary sequences of real-valued random variables satisfying weak dependence conditions including strong mixing in the sense of Rosenblatt (1956) [30] as a special case. Applications to unbounded functions of intermittent maps are given.

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1. Introduction

The almost sure invariance principle is a powerful tool in both probability and statistics. It says that the partial sums of random variables can be approximated by those of independent Gaussian random variables, and that the approximation error between the trajectories of the two processes is negligible compared to their size. More precisely, when $(X_i)_{i\geq 1}$ is a sequence of i.i.d. centered real valued random variables with a finite second moment, a sequence $(Z_i)_{i\geq 1}$ of i.i.d. centered

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Gaussian variables may be constructed in such a way that

$$\sup_{1\le k\le n}\left|\sum_{i=1}^{k}(X_i - Z_i)\right| = o(a_n) \quad \text{almost surely,} \tag{1.1}$$

where $(a_n)_{n\ge 1}$ is a nondecreasing sequence of positive reals tending to infinity. The first result of this type is due to Strassen [34] who obtained (1.1) with $a_n = (n \log\log n)^{1/2}$. To get smaller (a_n) additional information on the moments of X_1 is necessary. If $\mathbb{E}|X_1|^p < \infty$ for p in $]2, 4[$, by using the Skorohod embedding theorem, Breiman [4] showed that (1.1) holds with $a_n = n^{1/p}(\log n)^{1/2}$. He also proved that $a_n = n^{1/p}$ cannot be improved under the p-th moment assumption for any $p > 2$. The Breiman paper highlights the fact that there is a gap between the direct result and its converse when using the Skorohod embedding. This gap was later filled by Komlós et al. [16] for $p > 3$ and by Major [20] for p in $]2, 3]$: they obtained (1.1) with $a_n = n^{1/p}$ as soon as $\mathbb{E}|X_1|^p < \infty$ for any $p > 2$, using an explicit construction of the Gaussian random variables, based on quantile transformations.

There has been a great deal of work to extend these results to dependent sequences; see for instance [24,2,6,3,31,12,35,36,13,1,5], among others, for extensions of (1.1) under various dependence conditions.

In this paper, we are interested in the case of strictly stationary strongly mixing sequences. Recall that the strong mixing coefficient of Rosenblatt [30] between two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\alpha(\mathcal{F}, \mathcal{G}) = \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

For a strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$ of real valued random variables, and the σ-algebras $\mathcal{F}_0 = \sigma(X_i, i \le 0)$ and $\mathcal{G}_n = \sigma(X_i, i \ge n)$, then define

$$\alpha(0) = 1 \quad \text{and} \quad \alpha(n) = 2\alpha(\mathcal{F}_0, \mathcal{G}_n) \quad \text{for } n > 0. \tag{1.2}$$

Concerning the extension of (1.1) in the strong mixing setting, Rio [26] proved the following: assume that

$$\sum_{k=0}^{\infty}\int_0^{\alpha(k)} Q^2_{|X_0|}(u)\mathrm{d}u < \infty, \tag{1.3}$$

where $Q_{|X_0|}$ is given in Definition 2.1. Then the series $\mathbb{E}(X_0^2) + 2\sum_{k\ge 1}\mathbb{E}(X_0X_k)$ is convergent to a nonnegative real σ^2 and one can construct a sequence $(Z_i)_{i\ge 1}$ of zero mean i.i.d. Gaussian variables with variance σ^2 such that (1.1) holds true with $a_n = (n \log\log n)^{1/2}$. As shown in Theorem 3 of Rio [26], the condition (1.3) cannot be improved. Recently, Dedecker et al. [8] have proved that this result still holds if we replace the Rosenblatt strong mixing coefficients $\alpha(n)$ by the weaker coefficients defined in (2.1), provided that the underlying sequence is ergodic.

Still in the strong mixing setting, up to our knowledge, the best extension of the Komlós et al. results is due to Shao and Lu [32]. Applying the Skorohod embedding, they obtained the following result (see also Corollary 9.3.1 in [18]): let p in $]2, 4]$ and $r > p$. Assume that

$$\mathbb{E}(|X_0|^r) < \infty \quad \text{and} \quad \sum_{n\ge 1}(\alpha(n))^{(r-p)/(rp)} < \infty. \tag{1.4}$$