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Extremes of space–time Gaussian processes

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Abstract

Let $Z = \{Z_t(h); h \in \mathbb{R}^d, t \in \mathbb{R}\}$ be a space–time Gaussian process which is stationary in the time variable t. We study $M_n(h) = \sup_{t\in[0,n]} Z_t(s_n h)$, the supremum of Z taken over $t \in [0, n]$ and rescaled by a properly chosen sequence $s_n \to 0$. Under appropriate conditions on Z, we show that for some normalizing sequence $b_n \to \infty$, the process $b_n(M_n - b_n)$ converges as $n \to \infty$ to a stationary max-stable process of Brown–Resnick type. Using strong approximation, we derive an analogous result for the empirical process.

MSC: primary 60G15; 60G70; secondary 60G60; 60F17

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1. Introduction and statement of results

1.1. Introduction

Let X_i, $i \in \mathbb{N}$, be independent copies of a Gaussian process $\{X(h); h \in D\}$. We suppose that X has zero mean, unit variance, continuous sample paths, and is defined on $D \subset \mathbb{R}^d$, an open set containing the origin. Further, we suppose that the covariance function of X, $r^X(h_1, h_2) = \mathbb{E}[X(h_1)X(h_2)]$, satisfies the following condition: for some $\alpha \in (0, 2]$ and $c_\alpha > 0$,

(X1) $r^X(\varepsilon h_1, \varepsilon h_2) = 1 - c_\alpha|h_1 - h_2|^\alpha \varepsilon^\alpha + o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$, where the o-term is uniform in $h_1, h_2 \in D$.

Here, $|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^d$.

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The limiting properties, as $n \to \infty$, of the maximum of $X_1, \dots, X_n$, taken pointwise, were studied by Brown and Resnick [3] and Kabluchko et al. [13]. It was observed in [3] that in order to obtain a nontrivial limiting process, an additional spatial rescaling need to be introduced. We need normalizing sequences s_n and b_n defined by

$$s_n = \frac{1}{(2c_\alpha \log n)^{1/\alpha}}, \tag{1}$$

$$b_n = \sqrt{2\log n} - \frac{1}{\sqrt{2\log n}}\left(\frac{1}{2}\log\log n + \log(2\sqrt{\pi})\right). \tag{2}$$

Define a stochastic process $\{M_n(h); h \in s_n^{-1}D\}$, where $s_n^{-1}D$ denotes the set $\{s_n^{-1}h : h \in D\}$, by

$$M_n(h) = \max_{i=1,\dots,n} X_i(s_n h). \tag{3}$$

Then it follows from a more general result of [13, Theorem 17] that the process M_n^* defined by $M_n^*(h) = b_n(M_n(h) - b_n)$ converges as $n \to \infty$ to some nontrivial limiting process η_α weakly on $C(K)$, the space of continuous functions on any fixed compact set $K \subset \mathbb{R}^d$. If X is the Ornstein–Uhlenbeck process on $\mathbb{R}$ with covariance function $r^X(h_1, h_2) = \mathrm{e}^{-|h_1-h_2|}$, the above result is due to Brown and Resnick [3], the limit being η_1. Other particular cases, leading to the process η_2, were considered in [8,9]. Closely related results were obtained by Pickands [18,19] and Hüsler and Reiss [11]. Applications were given in [7,4].

The limiting process η_α will be called the Brown–Resnick process with parameter $\alpha \in (0, 2]$, and can be described as follows. Let $\{U_i\}_{i=1}^{\infty}$ be an enumeration of the points of a Poisson point process with intensity $\mathrm{e}^{-u}\mathrm{d}u$ on $\mathbb{R}$. Further, let W_i, $i \in \mathbb{N}$, be independent copies of a drifted (Lévy) fractional Brownian motion $\{W(h); h \in \mathbb{R}^d\}$ with

$$\mathrm{Cov}(W(h_1), W(h_2)) = |h_1|^\alpha + |h_2|^\alpha - |h_1 - h_2|^\alpha, \tag{4}$$

$$\mathbb{E}[W(h)] = -|h|^\alpha. \tag{5}$$

Then the Brown–Resnick process $\{\eta_\alpha(h); h \in \mathbb{R}^d\}$ is defined by

$$\eta_\alpha(h) = \max_{i \in \mathbb{N}}(U_i + W_i(h)). \tag{6}$$

The process η_α is stationary (although this is not evident from Eq. (6)), sample continuous, with unit Gumbel margins, see [13] for more properties.

1.2. Main result

In this paper, we study the limiting behavior of the supremum, taken over continuous time and considered as a function of space, of a space–time Gaussian process. Let us be more precise. Let $Z = \{Z_t(h); h \in D, t \in \mathbb{R}\}$ be a zero mean and unit variance sample continuous Gaussian process which is stationary in the time variable t. Here, D is an open subset of $\mathbb{R}^d$ containing 0. The covariance function of Z, $r_t(h_1, h_2) = \mathbb{E}[Z_s(h_1)Z_{s+t}(h_2)]$, does not depend on s by the time stationarity.

We suppose that the following conditions are satisfied for some $\alpha, \beta \in (0, 2]$ and $c_\alpha, c_\beta > 0$:

(Z1) $r_{\varepsilon^{1/\beta}t}(\varepsilon^{1/\alpha}h_1, \varepsilon^{1/\alpha}h_2) = 1 - (c_\alpha|h_1 - h_2|^\alpha + c_\beta|t|^\beta)\varepsilon + o(\varepsilon)$ as $\varepsilon \downarrow 0$, where the o-term is uniform as long as $h_1, h_2 \in D$ and t stays bounded.

(Z2) $r_t(h_1, h_2) < 1$ provided that $t \neq 0$, $h_1, h_2 \in D$.

(Z3) $r_t(h_1, h_2) = o(1/\log|t|)$ as $t \to \infty$ uniformly in $h_1, h_2 \in D$.