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Stochastic Processes and their Applications 121 (2011) 1982–2013
stochastic processes and their applications
www.elsevier.com/locate/spa

# Local time-space calculus for symmetric Lévy processes

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Received 12 May 2010; received in revised form 6 April 2011; accepted 14 May 2011
Available online 20 May 2011

**Abstract**

We construct a stochastic calculus with respect to the local time process of a symmetric Lévy process $X$ without Brownian component. The required assumptions on the Lévy process are satisfied by the symmetric stable processes with index in $(1, 2)$. Based on this construction, the explicit decomposition of $F(X_t, t)$ is obtained for $F$ continuous function admitting a Radon–Nikodym derivative $\frac{\partial F}{\partial t}$ and satisfying some integrability condition. This Itô formula provides, in particular, the precise expression of the martingale and the continuous additive functional present in Fukushima's decomposition.


*MSC:* 60G44; 60H05; 60J55; 60J65

*Keywords:* Stochastic calculus; Local time-space calculus; Itô formula; Lévy process; Symmetric stable process

## 1. Introduction and main results

For a given semimartingale $(X_t)_{t\geq 0}$ and any $\mathcal{C}^{2,1}$-function $F$ on $\mathbb{R} \times \mathbb{R}^+$, the Itô formula provides both an explicit expansion of $(F(X_t, t))_{t\geq 0}$ and its stochastic structure. Consider the case when $X$ is a Lévy process with characteristic triplet $(a, \sigma, \nu)$ which means that for any $t$ in $\mathbb{R}_+$ and $\xi$ in $\mathbb{R}$: $\mathbf{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}$, where: $\psi(\xi) = -ia\xi + \frac{\sigma^2}{2}\xi^2 + \int_{\mathbb{R}}(1 - e^{i\xi x} + i\xi x 1_{|x|\leq 1})\nu(\mathrm{d}x)$, $a \in \mathbb{R}, \sigma \in \mathbb{R}_+$ and $\nu$ is a measure in $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \frac{x^2}{1+x^2}\nu(\mathrm{d}x) < \infty$. The function $\psi$ is called the characteristic component of $X$ and $\nu$, the Lévy measure of $X$ (see Bertoin [2]). Denote by $\sigma B$ the Brownian component of $X$, then the Itô

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doi:10.1016/j.spa.2011.05.011

formula can be rewritten under the following form (see e.g., Ikeda and Watanabe [11]):

$$F(X_t, t) = F(X_0, 0) + M_t + A_t, \tag{1.1}$$

where $M$ is a local martingale and $A$ is an adapted process of bounded variation given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s)\mathrm{d}B_s + \int_0^t \int_{\{|y|\le 1\}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\}\tilde{\mu}_X(\mathrm{d}y, \mathrm{d}s)$$

$$A_t = \sum_{0<s\le t} \{F(X_s, s) - F(X_{s-}, s)\}1_{\{|\Delta X_s|>1\}} + \int_0^t \mathcal{A}F(X_s, s)\mathrm{d}s$$

where $\tilde{\mu}_X(\mathrm{d}y, \mathrm{d}s)$ denotes the compensated Poisson measure associated to the jumps of $X$, and $\mathcal{A}$ is the operator associated to $X$ defined by

$$\begin{aligned}\mathcal{A}G(x,t) = {} & \frac{\partial G}{\partial t}(x,t) + a\frac{\partial G}{\partial x}(x,s) + \frac{1}{2}\sigma^2\frac{\partial^2 G}{\partial x^2}(x,t) \\ & + \int_{\mathbb{R}} \left\{G(x+y,t) - G(x,t) - y\frac{\partial G}{\partial x}(x,t)\right\} 1_{(|y|<1)}\nu(\mathrm{d}y)\end{aligned} \tag{1.2}$$

for any function $G$ defined on $\mathbb{R} \times \mathbb{R}^+$, such that $\frac{\partial G}{\partial x}$, $\frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial x^2}$ exist as Radon–Nikodym derivatives with respect to the Lebesgue measure and the integral is well defined.

Many authors have succeeded in relaxing the conditions on $F$ to write extended versions of (1.1) (see for example Errami et al. [9], Eisenbaum [6] and Eisenbaum and Kyprianou [7]). Under the assumption that $X$ has a Brownian component (i.e. $\sigma \neq 0$), we have established in [8] an extended version of (1.1) that can be considered as *optimal* in the sense that it requires the sole condition of existence of locally bounded first order Radon–Nikodym derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial t}$. Under that condition, this version gives the explicit decomposition of $F(X_t, t)$ as the sum of a Dirichlet process and a bounded variation process.

Here we treat the case $\sigma = 0$. If we assume additionally that $X$ is symmetric (i.e. $a = 0$ and $\nu$ is symmetric), then according to Fukushima [10], we already know that for every continuous function $u$ in $\mathcal{W}$, the Dirichlet space of $X$, i.e.

$$\mathcal{W} = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} (u(x+y) - u(x))^2 \mathrm{d}x\,\nu(\mathrm{d}y) < \infty\right\},$$

$u(X)$ admits the following decomposition

$$u(X_t) = u(X_0) + M_t^u + N_t^u \tag{1.3}$$

where $M^u$ is a square-integrable martingale and $N^u$ is a continuous additive functional with 0-quadratic energy. Besides, for $\Phi$ in $\mathcal{C}^2(\mathbb{R})$, Chen et al. [4] give a decomposition of $\Phi(u(X))$ in terms of $M^u$ and $N^u$.

In this paper we write an extension of (1.3) to time-space functions and give the explicit expression of the corresponding terms. In particular, the explicit expression of the processes $M^u$ and $N^u$ involved in (1.3) are obtained.

These results, precisely presented below, require two additional assumptions on $X$. The first one is the existence of local times for $X$ considered as a Markov process, i.e., a jointly measurable family $\{(L_t^x)_{t\ge 0}, x \in \mathbb{R}\}$ of positive additive functionals such that for each $x$, the measure $\mathrm{d}L_t^x$ is supported by the set $\{t \ge 0 : X_t = x\}$ and satisfying for every Borel-measurable function