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Critical point and percolation probability in a long range site percolation model on $\mathbb{Z}^d$

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Abstract

Consider an independent site percolation model with parameter $p \in (0, 1)$ on $\mathbb{Z}^d$, $d \geq 2$, where there are only nearest neighbor bonds and long range bonds of length k parallel to each coordinate axis. We show that the percolation threshold of such a model converges to $p_c(\mathbb{Z}^{2d})$ when k goes to infinity, the percolation threshold for ordinary (nearest neighbor) percolation on $\mathbb{Z}^{2d}$. We also generalize this result for models whose long range bonds have several lengths.

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1. Introduction and notation

Let $G = (\mathbb{V}, \mathbb{E})$ be a graph with a countably infinite vertex set $\mathbb{V}$. Consider the Bernoulli site percolation model on G; with each site $v \in \mathbb{V}$ we associate a Bernoulli random variable $X(v)$, which takes the values 1 and 0 with probability p and $1 - p$ respectively. This can be done considering the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where $\Omega = \{0, 1\}^{\mathbb{V}}$, $\mathcal{F}$ is the σ-algebra

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generated by the cylinder sets in Ω and $\mathbb{P}_p = \prod_{v \in \mathbb{V}} \mu(v)$ is the product of Bernoulli measures with parameter p, in which the configurations $\{X(v), v \in \mathbb{V}\}$ occur. We denote a typical element of Ω by ω. When $X(v) = 1$ (respectively, $X(v) = 0$) we say that v is "open" (respectively, "closed").

Given two vertices v and u, we say that v and u are connected in the configuration ω if there exists a finite path $\langle v = v_0, v_1, \dots, v_n = u \rangle$ of open vertices in $\mathbb{V}$ such that $v_i \neq v_j, \ \forall i \neq j$ and $\langle v_i, v_{i+1} \rangle$ belongs to $\mathbb{E}$ for all $i = 0, 1, \dots, n-1$. We will use the short notation $\{v \leftrightarrow u\}$ to denote the set of configurations where u and v are connected.

Given the vertex v, the cluster of v in the configuration ω is the set $C_v(\omega) = \{u \in \mathbb{V}; v \leftrightarrow u \text{ on } \omega\}$. We say that the vertex v percolates when the cardinality of $C_v(\omega)$ is infinite; we will use the following standard notation: $\{v \leftrightarrow \infty\} \equiv \{\omega \in \Omega; \#C_v(\omega) = \infty\}$. Fixing some vertex v, we define the percolation probability of the vertex v as the function $\theta_v(p) : [0, 1] \mapsto [0, 1]$ with $\theta_v(p) = \mathbb{P}_p(v \leftrightarrow \infty)$.

From now on, the vertex set $\mathbb{V}$ will be $\mathbb{Z}^d, \ d \geq 2$, and for each positive integer k we define

$$\mathbb{E}_k = \{\langle (v_1, \dots, v_d), (u_1, \dots, u_d) \rangle \in \mathbb{V} \times \mathbb{V}; \exists! i \in \{1, \dots, d\} \text{ such that } |v_i - u_i| = k \text{ and } v_j = u_j, \forall j \neq i\}.$$

Let us define the graph $G^k = (\mathbb{V}, \mathbb{E}_1 \cup \mathbb{E}_k)$, that is, G^k is $\mathbb{Z}^d$ equipped with nearest neighbor bonds and long range bonds with length k parallel to each coordinate axis. Observe that G^k is a transitive graph; hence the function $\theta_v(p)$ does not depend on v and we write only $\theta^k(p)$ to denote $\mathbb{P}_p(0 \leftrightarrow \infty)$ for any transitive graph.

The simplest version of our main result (see Theorem 1) states that $p_c(G^k)$ converges to $p_c(\mathbb{Z}^{2d})$ when k goes to infinity. The main motivation for studying this question is that we believe that the Conjecture 1 stated below can shed some light on the truncation problem for long range percolation. This problem, proposed by Andjel, is the following:

On $\mathbb{Z}^d, \ d \geq 2$, consider the complete graph $G = (\mathbb{Z}^d, \mathbb{E})$, that is for all $v, u \in \mathbb{Z}^d$ we have that $\langle u, v \rangle \in \mathbb{E}$. For each bond $\langle u, v \rangle \in \mathbb{E}$ we define its length as $\|u - v\|_1$. Given a sequence $(p_n \in [0, 1), n \in \mathbb{N})$ consider an independent bond percolation model where each bond whose length is n will be open with probability p_n. Assume that $\sum_{n \in \mathbb{N}} p_n = \infty$; by the Borel–Cantelli lemma, the origin will percolate to infinity with probability 1. The general and still open truncation question is the following: is it true that there exists some sufficiently large but finite integer K such that the origin in the truncated processes, obtained by deleting (or closing) all long range bonds whose length are bigger than K, still percolates to infinity with positive probability?

The general question is still open; see [4] for a more detailed discussion.

2. The main result

Given a positive integer n, define the n-vector $\overrightarrow{k} = (k_1, \dots, k_n)$, where $k_i \in \{2, 3, \dots\}, \forall i = 1, \dots, n$. We define the graph $G^{\overrightarrow{k}}$ as $(\mathbb{Z}^d, \mathbb{E}_1 \cup (\cup_{i=1}^n \mathbb{E}_{k_1 \times \cdots \times k_i}))$. Observe that when $n = 1$ and $k_1 = k$, the graph $G^{\overrightarrow{k}}$ is the graph G^k defined above. That is, $G^{\overrightarrow{k}}$ is $\mathbb{Z}^d$ decorated with all bonds parallel to each coordinate axis with lengths $1, k_1, k_1 \times k_2, \dots, k_1 \times k_2 \times \cdots \times k_n$.

From now on, we will use the notation $S^{\overrightarrow{k}}$ to denote the $d(n+1)$-dimensional slab graph where the vertex set is $(\mathbb{Z} \times \prod_{i=1}^n \{0, 1, \dots, k_{n-i+1} - 1\})^d$ and $S^{\overrightarrow{k}}$ is equipped with only nearest neighbor bonds.

The aim of this note is to prove that the percolation function of the graph $G^{\overrightarrow{k}}$ is bounded between the percolation functions of the slab $S^{\overrightarrow{k}}$ and of $\mathbb{Z}^{d(n+1)}$. More precisely, we have the