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Least squares estimator for Ornstein–Uhlenbeck processes driven by α-stable motions

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Abstract

We study the problem of parameter estimation for generalized Ornstein–Uhlenbeck processes driven by α-stable noises, observed at discrete time instants. Least squares method is used to obtain an asymptotically consistent estimator. The strong consistency and the rate of convergence of the estimator have been studied. The estimator has a higher order of convergence in the general stable, non-Gaussian case than in the classical Gaussian case.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$. Let $\{Z_t, t \geq 0\}$ be a standard symmetric α-stable Lévy motion. For technical reasons, we assume that $1 < \alpha < 2$. The generalized Ornstein–Uhlenbeck process $\{X_t, t \geq 0\}$, starting from $x \in \mathbb{R}$ is defined as the unique solution to the following linear stochastic differential equation (SDE)

$$\mathrm{d}X_t = -\theta_0 X_t \mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x. \tag{1.1}$$

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Assume that this process is observed at some discrete time instants $\{t_i = ih, i = 0, 1, 2, \ldots\}$, but the value of θ_0 is unknown. The purpose of this paper is to study the least squares estimator (LSE) for the true value θ_0 based on the sampling data $(X_{t_i})_{i=0}^n$.

In the case of diffusion processes driven by Brownian motions, a popular method is the maximum likelihood estimator (MLE) based on the Girsanov density (see [1]). It is asymptotically equivalent to the least squares estimator. For the LSE the convergence in probability is proved in [2,3], the strong consistency is studied in [4], and the asymptotic distribution was studied in [5]. For a more recent comprehensive discussion, we refer to [6,7] and the references therein. For MLE based on discrete observations, see for example [8].

Recently there has been a growing interest in parameter estimation for stochastic processes driven by Lévy processes with finite moments due to its promising applications for example to finance. Substantial progress has been made. The asymptotic normality of the LSE and MLE for pure jump process is studied in [9,10]. The paper [11] dealt with the consistency and asymptotic normality when the driving process is a zero-mean adapted process (including Lévy process) with finite moments. However, when the driving processes are α-stable Lévy motions there has been no study yet due to the infinite variance property of α-stable processes.

The main focus of this paper is the study of the strong consistency and asymptotic distributions of the LSE for generalized O–U processes satisfying the SDE (1.1). Our results are analogues of the LSE and the Yule–Walker estimator for ARMA models driven by a sequence of i.i.d. random variables in the domain of attraction of a stable law (see Davis and Resnick [12]). Other related estimators such as M-estimator and the Whittle estimator can be found in [13,14].

To obtain the LSE, we introduce the following *contrast function*

$$\rho_n(\theta) = \rho_n\left(\theta; (X_{t_i})_{i=0}^n\right) = \sum_{i=1}^n \left|X_{t_i} - X_{t_{i-1}} + \theta X_{t_{i-1}} \cdot \Delta t_{i-1}\right|^2. \tag{1.2}$$

Then the LSE $\hat{\theta}_n$ is defined as $\hat{\theta}_n = \arg\min_{\theta>0} \rho_n(\theta)$, which can be explicitly represented as

$$\hat{\theta}_n = -\frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) X_{t_{i-1}}}{h \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.3}$$

The equation can be solved explicitly so that we can represent the LSE $\hat{\theta}_n$ as

$$\hat{\theta}_n = \frac{1 - \mathrm{e}^{-\theta_0 h}}{h} - \frac{\sum_{i=1}^n X_{t_{i-1}} \cdot \int_{t_{i-1}}^{t_i} \mathrm{e}^{-\theta_0 (t_i - s)} \mathrm{d}Z_s}{h \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{1.4}$$

In this paper, high frequency ($h \to 0$) asymptotic of the LSE $\hat{\theta}_n$ is considered in the ergodic case ($\theta_0 > 0$). Our goal is to prove that $\hat{\theta}_n \to \theta_0$ almost surely and to establish the rate of convergence $(\frac{\log n}{nh})^{1/\alpha}$. This rate is considerably faster than in the Brownian motion case.

If the processes can be observed continuously, a trajectory fitting method combined with weighted least squares technique is discussed in [15].