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Formulas for stopped diffusion processes with stopping times based on drawdowns and drawups

Libor Pospisil, Jan Vecer*, Olympia Hadjiliadis

Department of Statistics, Columbia University, 1255 Amsterdam Avenue, NY 10027, USA
Department of Mathematics, Brooklyn College (C.U.N.Y.), Ingersoll Hall, NY 11209, USA

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Abstract

This paper studies drawdown and drawup processes in a general diffusion model. The main result is a formula for the joint distribution of the running minimum and the running maximum of the process stopped at the time of the first drop of size a. As a consequence, we obtain the probabilities that a drawdown of size a precedes a drawup of size b and vice versa. The results are applied to several examples of diffusion processes, such as drifted Brownian motion, Ornstein–Uhlenbeck process, and Cox–Ingersoll–Ross process.

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Keywords: Diffusion process; Drawdowns and drawups; Stopping time

1. Introduction

In this article, we study properties of a general diffusion process $\{X_t\}$ stopped at the first time when its drawdown attains a certain value a. Let us denote this time as $T_D(a)$. The drawdown of a process is defined as the current drop of the process from its running maximum. We present two main results here. First, we derive the joint distribution of the running minimum and the running maximum stopped at $T_D(a)$. Second, we calculate the probability that a drawdown of

* Corresponding author at: Department of Statistics, Columbia University, 1255 Amsterdam Avenue, NY 10027, USA. Tel.: +1 212 851 2145; fax: +1 212 851 2164.
E-mail addresses: lp2185@columbia.edu (L. Pospisil), vecer@stat.columbia.edu (J. Vecer), ohadjiliadis@brooklyn.cuny.edu (O. Hadjiliadis).

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size a precedes a drawup of size b, where the drawup is defined as the increase of $\{X_t\}$ over the running minimum. All formulas are expressed in terms of the drift function, the volatility function, and the initial value of $\{X_t\}$. In addition to the main theorems, this paper contains other results that help us to understand the behavior of diffusion processes better. For example, we relate the probability that the drawup process stopped at $T_D(a)$ is zero to the expected running minimum stopped at $T_D(a)$.

We apply the results to several examples of diffusion processes: drifted Brownian motion, Ornstein–Uhlenbeck process (OU), and Cox–Ingersoll–Ross process (CIR). These examples play important roles in change point detection and in finance. We also discuss how the results presented in this paper are related to the problem of quickest detection and identification of two-sided changes in the drift of general diffusion processes.

Our results extend several theorems stated and proved in [1–3]. These results include the distribution of a diffusion process stopped at the first time it hits either a lower or an upper barrier, and the distribution of the running maximum of a diffusion process stopped at time $T_D(a)$. The formulas for a drifted Brownian motion presented here coincide with the results in [4]. The approach used in [4] is based on a calculation of the expected first passage times of the drawdown and drawup processes to levels a and b. However, while this approach applies to a drifted Brownian motion, it cannot be extended to a general diffusion process. In this paper, we derive the joint distribution of the running maximum and minimum stopped at $T_D(a)$, which can be obtained for a general diffusion process. Subsequently, we use this result to calculate the probability that a drawdown precedes a drawup.

Properties of drawdown and drawup processes are of interest in change point detection, where the goal is to test whether an abrupt change in a parameter of a dynamical system has occurred. Drawdowns and drawups of the likelihood ratio process serve as test statistics for hypotheses about the change point. Details can be found, for example, in [5–7].

The concept of a drawdown has been also been studied in applied probability and in finance. The underlying diffusion process usually represents a stock index, an exchange rate, or an interest rate. Some characteristics of its drawdown, such as the expected maximum drawdown, can be used to measure the downside risks of the corresponding market. The distribution of the maximum drawdown of a drifted Brownian motion was determined in [8]. Cherny and Dupire [9] derived the distribution of a local martingale and its maximum at the first time when the corresponding range process attains value a. Salminen and Vallois [10] derived the joint distribution of the maximum drawdown and the maximum drawup of a Brownian motion up to an independent exponential time. Vecer [11] related the expected maximum drawdown of a market to directional trading. Several authors, such as Grossman and Zhou [12], Cvitanic and Karatzas [13], and Chekhlov et al. [14], discussed the problem of portfolio optimization with drawdown constraints. Meilijson [15] used stopping time $T_D(a)$ to solve an optimal stopping problem based on a drifted Brownian motion and its running maximum. Obloj and Yor [16] studied properties of martingales with representation $H(M_t, \bar{M}_t)$, where M_t is a continuous local martingale and $\bar{M}_t$ its supremum up to time t. Nikeghbali [17] associated the Skorokhod stopping problem with a class of submartingales which includes drawdown processes of continuous local martingales.

This paper is structured in the following way: notation and assumptions are introduced in Section 2. In Section 3, we derive the joint distribution of the running maximum and the running minimum stopped at the first time that the process drops by a certain amount (Theorem 3.1), and in Section 4, we calculate the probability that a drawdown of size a will precede a drawup of size b (Theorems 4.1 and 4.2). Special cases, such as drifted Brownian motion, Ornstein–Uhlenbeck process, Cox–Ingersoll–Ross process, are discussed in Section 5. The relevance of the result in