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Stochastic Processes and their Applications 121 (2011) 673–700
stochastic processes and their applications
www.elsevier.com/locate/spa

Ergodicity of the 3D stochastic Navier–Stokes equations driven by mildly degenerate noise

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Received 7 July 2009; received in revised form 3 October 2010; accepted 15 November 2010
Available online 21 November 2010

Abstract

We prove that any Markov solution to the 3D stochastic Navier–Stokes equations driven by a mildly degenerate noise (i.e. all but finitely many Fourier modes are forced) is uniquely ergodic. This follows by proving strong Feller regularity and irreducibility.

MSC: primary 76D05; secondary 60H15; 35Q30; 60H30; 76M35

Keywords: Stochastic Navier–Stokes equations; Martingale problem; Markov selections; Continuous dependence; Ergodicity; Degenerate noise; Malliavin calculus

1. Introduction

The well-posedness of three-dimensional Navier–Stokes equations is still an open problem, in both the deterministic and stochastic cases (see [8] for a general introduction to the deterministic problem and [9] for the stochastic one). Although the existence of global weak solutions has been proven in both cases [20,10], the uniqueness is still unknown. Inspired by the Hadamard definition of well-posedness for Cauchy problems, it is natural to ask whether there are ways to find a good selection among the weak solutions in order to obtain additional properties, such as Markovianity and continuity with respect to the initial data.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.11.010

Da Prato and Debussche proved in [4] that there exists a continuous selection (i.e. the selection is strong Feller) with unique invariant measure by studying the Kolmogorov equation associated with the stochastic Navier–Stokes equations. Later Debussche and Odasso [6] proved that this selection is also Markovian and Odasso [22] proved exponential mixing. A different and slightly more general approach to Markov solutions, which includes the cases of degenerate noise and even deterministic equations, was introduced in [14] (see also [2] for an application to a different model of the same method). Under the assumption of non-degeneracy and regularity of the covariance, the authors also proved that every Markov solution is strong Feller. Under the same assumptions every such dynamics is uniquely ergodic and exponentially mixing [24]. However, both approaches rely on the non-degeneracy of the driving noise to obtain the strong Feller property, and consequently ergodicity.

The strong Feller property and ergodicity of SPDEs driven by *degenerate* noise have been intensively studied in recent years (see for instance [7,29,16,23]). For the Navier–Stokes equations in dimension 2 there are several results on ergodicity (see for instance [11,3,18,19]), among which the most remarkable one is that of Hairer and Mattingly [17]. They prove that the 2D stochastic dynamics has a unique invariant measure as long as the noise forces at least two linearly independent Fourier modes. In this respect the three-dimensional case is still open (only partial results are known; see the aforementioned [4,24,14], and see also [23]) and this paper tries to partly fill this gap. More precisely, we will study the three-dimensional Navier–Stokes equations

$$\begin{cases} \dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \dot{\eta}, \\ \operatorname{div} u = 0, \\ u(0) = x, \end{cases} \tag{1.1}$$

on the torus $[0, 2\pi]^3$ with periodic boundary conditions and forced by a Gaussian noise $\dot{\eta}$. We assume that *all except finitely many Fourier modes* are driven by the noise, and prove that any Markov solution to the problem is strong Feller and ergodic.

Essentially, our approach combines the Malliavin calculus developed in [7] and the *weak–strong uniqueness* principle of [14]. Comparing with well-posed problems, the dynamics here exists only in the weak martingale sense and the standard tools of stochastic analysis are not available. Hence, the computations are made on an approximate cutoff dynamics (see Section 3), which equals any dynamics up to a small time. On the other hand, due to the degeneracy of the noise, the Bismut–Elworthy–Li formula cannot be directly applied to prove the strong Feller property. To fix this problem, we divide the dynamics into high and low frequencies, applying the formula only to the dynamics of high modes (thanks to the essential non-degeneracy of the noise).

Finally, we remark that, at least with the approach presented here, general results such as the truly hypoelliptic case in [17] seem to be hardly achievable. Here (as well as in [14]) the strong Feller property is essential for propagating smoothness from small times (where trajectories are regular with high probability) to all times. To understand how to study the general case, the second author [1] has replicated the results of this paper using the approach, via the Kolmogorov equation, originally used in [4].

The paper is organized as follows. Section 2 gives a detailed description of the problem, the assumptions on the noise and the main results (Theorems 2.3 and 2.5). Section 3 contains the proof of strong Feller regularity, while Section 4 applies Malliavin calculus to prove the crucial Lemma 3.4. Section 5 shows the irreducibility of the dynamics; the Appendix contains additional details and the proofs of some technical results.