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Stochastic Processes and their Applications 121 (2011) 1290–1314
stochastic processes and their applications
www.elsevier.com/locate/spa

On the local time of random walk on the 2-dimensional comb

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Received 21 June 2010; received in revised form 20 January 2011; accepted 24 January 2011
Available online 1 February 2011

Abstract

We study the path behaviour of general random walks, and that of their local times, on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal edges off the x-axis. We prove strong approximation results for such random walks and also for their local times. Concentrating mainly on the latter, we establish strong and weak limit theorems, including Strassen-type laws of the iterated logarithm, Hirsch-type laws, and weak convergence results in terms of functional convergence in distribution.

MSC: primary 60F17; 60G50; 60J65; secondary 60F15; 60J10

Keywords: Random walk; 2-dimensional comb; Strong approximation; 2-dimensional Wiener process; Local time; Laws of the iterated logarithm; Iterated Brownian motion

1. Introduction and main results

In this paper we continue our study of a simple random walk $\mathbf{C}(n)$ on the 2-dimensional comb lattice $\mathbb{C}^2$ that is obtained from $\mathbb{Z}^2$ by removing all horizontal lines off the x-axis (cf. [16]).

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doi:10.1016/j.spa.2011.01.009

A formal way of describing a simple random walk $\mathbf{C}(n)$ on the above 2-dimensional comb lattice $\mathbb{C}^2$ can be formulated via its transition probabilities as follows: for $(x, y) \in \mathbb{Z}^2$

$$\mathbf{P}(\mathbf{C}(n+1) = (x, y \pm 1) \mid \mathbf{C}(n) = (x, y)) = \frac{1}{2}, \quad \text{if } y \neq 0, \tag{1.1}$$

$$\begin{aligned} \mathbf{P}(\mathbf{C}(n+1) &= (x \pm 1, 0) \mid \mathbf{C}(n) = (x, 0)) \\ &= \mathbf{P}(\mathbf{C}(n+1) = (x, \pm 1) \mid \mathbf{C}(n) = (x, 0)) = \frac{1}{4}. \end{aligned} \tag{1.2}$$

The coordinates of the just defined vector valued simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ are denoted by $C_1(n), C_2(n)$, i.e., $\mathbf{C}(n) := (C_1(n), C_2(n))$.

A compact way of describing the just introduced transition probabilities for this simple random walk $\mathbf{C}(n)$ on $\mathbb{C}^2$ is via defining

$$p(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n+1) = \mathbf{v} \mid \mathbf{C}(n) = \mathbf{u}) = \frac{1}{\deg(\mathbf{u})}, \tag{1.3}$$

for locations $\mathbf{u}$ and $\mathbf{v}$ that are neighbors on $\mathbb{C}^2$, where $\deg(\mathbf{u})$ is the number of neighbors of $\mathbf{u}$, otherwise $p(\mathbf{u}, \mathbf{v}) := 0$. Consequently, the non-zero transition probabilities are equal to $1/4$ if $\mathbf{u}$ is on the horizontal axis and they are equal to $1/2$ otherwise.

This and related models have been studied intensively in the literature and have a number of applications in various problems in physics. See, for example, [1–3,12,23,25,38,43,44], and the references in these papers. It was observed that the second component $C_2(n)$ behaves like ordinary Brownian motion, but the first component $C_1(n)$ exhibits some anomalous subdiffusion property of order $n^{1/4}$. Zahran [43] and Zahran et al. [44] applied the Fokker–Planck equation to describe the properties of the comb-like model. Weiss and Havlin [42] derived the asymptotic form for the probability that $\mathbf{C}(n) = (x, y)$ by appealing to a central limit argument. Bertacchi and Zucca [8] obtained space–time asymptotic estimates for the n-step transition probabilities $p^{(n)}(\mathbf{u}, \mathbf{v}) := \mathbf{P}(\mathbf{C}(n) = \mathbf{v} \mid \mathbf{C}(0) = \mathbf{u})$, $n \geq 0$, from $\mathbf{u} \in \mathbb{C}^2$ to $\mathbf{v} \in \mathbb{C}^2$, when $\mathbf{u} = (2k, 0)$ or $(0, 2k)$ and $\mathbf{v} = (0, 0)$. Using their estimates, they concluded that, if k/n goes to zero with a certain speed, then $p^{(2n)}((2k, 0), (0, 0))/p^{(2n)}((0, 2k), (0, 0)) \to 0$, as $n \to \infty$, an indication that suggests that the particle in this random walk spends most of its time on some tooth of the comb. Bertacchi [7] noted that a Brownian motion is the right object to approximate $C_2(\cdot)$, but for the first component $C_1(\cdot)$ the right object is a Brownian motion time-changed by the local time of the second component. More precisely, Bertacchi [7] on defining the continuous time process $\mathbf{C}(nt) = (C_1(nt), C_2(nt))$ by linear interpolation, established the following remarkable joint weak convergence result.

Theorem A. *For the $\mathbb{R}^2$ valued random elements $\mathbf{C}(nt)$ of $C[0, \infty)$ we have*

$$\left(\frac{C_1(nt)}{n^{1/4}}, \frac{C_2(nt)}{n^{1/2}};\ t \geq 0 \right) \xrightarrow{\text{Law}} (W_1(\eta_2(0, t)), W_2(t);\ t \geq 0), \quad n \to \infty, \tag{1.4}$$

where W_1, W_2 are two independent Brownian motions and $\eta_2(0, t)$ is the local time process of W_2 at zero, and $\xrightarrow{\text{Law}}$ denotes weak convergence on $C([0, \infty), \mathbb{R}^2)$ endowed with the topology of uniform convergence on compact subsets.

Here, and throughout as well, $C(I, \mathbb{R}^d)$, respectively $D(I, \mathbb{R}^d)$, stand for the space of $\mathbb{R}^d$-valued, $d = 1, 2$, continuous, respectively *càdlàg*, functions defined on an interval $I \subseteq [0, \infty)$. $\mathbb{R}^1$ will be denoted by $\mathbb{R}$ throughout.