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Riesz transform and integration by parts formulas for random variables

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Abstract

We use integration by parts formulas to give estimates for the L^p norm of the Riesz transform. This is motivated by the representation formula for conditional expectations of functionals on the Wiener space already given in Malliavin and Thalmaier (2006) [13]. As a consequence, we obtain regularity and estimates for the density of non-degenerated functionals on the Wiener space. We also give a semi-distance which characterizes the convergence to the boundary of the set of the strict positivity points for the density.

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Keywords: Riesz transform; Integration by parts; Malliavin calculus; Sobolev spaces

1. Introduction

The starting point of this paper is the representation theorem for densities and conditional expectations of random variables based on the Riesz transform, recently given by Malliavin and Thalmaier in [13]. Let us recall it. Let F and G denote random variables taking values on $\mathbb{R}^d$ and $\mathbb{R}$ respectively and consider the following integration by parts formula: there exist some integrable random variables $H_i(F, G)$ such that for every test function $f \in C_c^\infty(\mathbb{R}^d)$

$$IP_i(F, G) \quad \mathbb{E}(\partial_i f(F)G) = -\mathbb{E}(f(F)H_i(F, G)), \quad i = 1, \ldots, d.$$

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Malliavin and Thalmaier proved that if $IP_i(F,1), i = 1,\ldots,d$ hold and the law of F has a continuous density p_F, then

$$p_F(x) = -\sum_{i=1}^{d} \mathbb{E}(\partial_i Q_d(F-x)H_i(F,1))$$

where Q_d denotes the Poisson kernel on $\mathbb{R}^d$, that is the fundamental solution of the Laplace operator. Moreover, they proved also that if $IP_i(F,G), i = 1,\ldots,d$, a similar representation formula holds also for the conditional expectation of G with respect to F. The interest of Malliavin and Thalmaier in these representations come from numerical reasons – this allows one to simplify the computation of densities and conditional expectations using a Monte Carlo method. This is crucial in order to implement numerical algorithms for solving non-linear PDE's or optimal stopping problems – for example for pricing American options. But there is a difficulty: the variance of the estimators produced by such a representation formula is infinite. Roughly speaking, this comes from the blowing up of the Poisson kernel around zero: $\partial_i Q_d \in L^p$ for $p < d/(d-1)$, so that $\partial_i Q_d \notin L^2$ for every $d \geq 2$. So estimates of $\mathbb{E}(|\partial_i Q_d(F-x)|^p)$ are crucial in this framework and this is the central point of interest in our paper. In [8,9], Kohasu-Higa and Yasuda proposed a solution to this problem using some cut off arguments. And in order to find the optimal cut off level they used the estimates of $\mathbb{E}(|\partial_i Q_d(F-x)|^p)$ which we prove in this paper (actually, they used a former version given in the preprint [2]).

So our central result concerns estimates of $\mathbb{E}(|\partial_i Q_d(F-x)|^p)$. It turns out that, in addition to the interest in numerical problems, such estimates represent a suitable instrument in order to obtain regularity of the density of functionals on the Wiener space—for which Malliavin calculus produces integration by parts formulas. Before going further let us mention that one may also consider integration by parts formulas of higher order, that is

$$IP_\alpha(F,G) \quad E(\partial_\alpha f(F)) = E(f(F)H_\alpha(F,G))$$

where $\alpha = (\alpha_1,\ldots,\alpha_k)$. We say that an integration by parts formula of order k holds if this is true for every $\alpha \in \{1,\ldots,d\}^k$. Now, a first question is: which is the order k of integration by parts that one needs in order to prove that the law of F has a continuous density p_F? If one employs a Fourier transform argument (see [14]) or the representation of the density by means of the Dirac function (see [1]) then one needs d integration by parts if $F \in \mathbb{R}^d$. In [11] Malliavin proves that integration by parts of order one is sufficient in order to obtain a continuous density, the dimension d does not matter (he employs some harmonic analysis arguments). A second problem concerns estimates of the density p_F (and of its derivatives) and such estimates involve the L^p norms of the weights $H_\alpha(F,1)$. In the approach using the Fourier transform or the Dirac function, $\|H_\alpha(F,1)\|_p, |\alpha| \leq d$ are involved if one estimates $\|p_F\|_\infty$. But in [15] Shigekawa obtains estimates of $\|p_F\|_\infty$ depending only on $\|H_i(F,1)\|_p$, so on the weights of order one (and similarly for derivatives). In order to do it, he needs some Sobolev inequalities that he proves using a representation formula based on the Green function and some estimates of modified Bessel functions. Our program and our results are similar but the instrument used in our paper is the Riesz transform and the estimates of the Poisson kernel mentioned above.

Let us be more precise. Notice that $IP_i(F,G)$ may also be written as

$$IP_i(F,G) \quad \int \partial_i f(x)g(x)\mu_F(dx) = -\int f(x)\partial_i^{\mu_F} g(x)\mu_F(dx)$$