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Stochastic Processes and their Applications 120 (2010) 2078–2099
stochastic processes and their applications
www.elsevier.com/locate/spa

Rate of escape and central limit theorem for the supercritical Lamperti problem

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Received 13 November 2009; received in revised form 19 May 2010; accepted 9 June 2010
Available online 16 June 2010

Abstract

The study of discrete-time stochastic processes on the half-line with mean drift at x given by $\mu_1(x) \to 0$ as $x \to \infty$ is known as *Lamperti's problem*. We give sharp almost-sure bounds for processes of this type in the case where $\mu_1(x)$ is of order $x^{-\beta}$ for some $\beta \in (0, 1)$. The bounds are of order $t^{1/(1+\beta)}$, so the process is super-diffusive but sub-ballistic (has zero speed). We make minimal assumptions on the moments of the increments of the process (finiteness of $(2 + 2\beta + \varepsilon)$-moments for our main results, so fourth moments certainly suffice) and do not assume that the process is time-homogeneous or Markovian. In the case where $x^\beta \mu_1(x)$ has a finite positive limit, our results imply a strong law of large numbers, which strengthens and generalizes earlier results of Lamperti and Voit. We prove an accompanying central limit theorem, which appears to be new even in the case of a nearest-neighbour random walk, although our result is considerably more general. This answers a question of Lamperti. We also prove transience of the process under weaker conditions than those that we have previously seen in the literature. Most of our results also cover the case where $\beta = 0$. We illustrate our results with applications to birth-and-death chains and to multi-dimensional non-homogeneous random walks.

MSC: 60G07; 60J10; 60F15; 60F05; 60G42

Keywords: Lamperti's problem; Almost-sure bounds; Law of large numbers; Central limit theorem; Birth-and-death chain; Transience; Inhomogeneous random walk

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2010.06.004

1. Introduction

In a pioneering series of papers [14–16] published in the early 1960s, Lamperti systematically studied how the asymptotic behaviour of a non-negative real-valued discrete-time stochastic process with asymptotically zero drift is governed by the (first two) moment functions of its increments. In the last two decades, there has been renewed interest in Lamperti's problem and in particular in its applications to studying the behaviour of complicated multi-dimensional processes (see e.g. [8,18]). A special case of Lamperti's problem supported on $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ is that of asymptotically-zero-drift birth-and-death chains, for which exact calculations are often possible (using for instance Karlin–McGregor theory [4,12,13]); although classically well-studied, there has been recent renewed interest in such birth-and-death chains (see e.g. [5,6]), particularly in the context of modelling random polymers (see e.g. [3,9]). The study of continuous-time analogues of the general Lamperti problem seems to have begun only recently: see e.g. [7].

Let us describe Lamperti's problem informally. Consider a stochastic process $X = (X_t)_{t \in \mathbb{Z}^+}$ on $[0, \infty)$. For now, suppose that X is a time-homogeneous Markov process (that is, a Markov process with stationary transition probabilities) and that its increment moment functions

$$\mu_k(x) = \mathbf{E}[(X_{t+1} - X_t)^k \mid X_t = x] \tag{1.1}$$

are well defined for $k \geq 0$; one way to ensure this is to impose a uniform bound on the increments. (We will relax all of these conditions shortly.) Lamperti's problem is to determine how the asymptotic behaviour of X depends upon μ_1 and μ_2.

Under mild regularity conditions, the behaviour of X is rather standard when, outside some bounded set, $\mu_1(x) \equiv 0$ (the zero-drift case) or $\mu_1(x)$ is uniformly bounded to one side of 0. Roughly speaking, in the zero-drift case X behaves like a simple symmetric random walk and is null-recurrent, in the case of uniformly negative drift X is positive-recurrent with exponentially decaying stationary distribution, and in the case of uniformly positive drift X is transient with positive speed (i.e., ballistic).

This motivates the study of the *asymptotically-zero-drift* regime, in which $\mu_1(x) \to 0$ as $x \to \infty$, to investigate phase transitions. It turns out that there is a rich spectrum of possible behaviours of X, governed by μ_1 and μ_2; we mention heavy-tailed positive-recurrence, transience with sub-linear rate of escape (diffusive and super-diffusive motion both being possible), weak convergence to a Bessel process, and so on.

Results of Lamperti [14,16] imply that from the point of view of the recurrence classification of X, the case where $|\mu_1(x)|$ is of order x^{-1} and $\mu_2(x)$ is of order 1 is critical. In the present paper we are interested in the *supercritical* case where $\mu_1(x)$ is positive and of order $x^{-\beta}$, $\beta \in (0, 1)$. Here, under mild conditions, transience is assured: our primary interest is to quantify this transience by studying the *rate of escape* and accompanying second-order behaviour.

As well as being of interest in their own right, stochastic processes on the half-line with mean drift asymptotically zero are important for the study of multi-dimensional processes by the method of Lyapunov-type functions (see e.g. [8]). In this context, it is particularly desirable to work in some generality without imposing, for instance, assumptions of the Markov property, a countable state-space, or uniformly bounded increments. Thus we work in more generality than the model outlined informally above. To start with, the assumption on uniformly bounded increments can be relaxed, and replaced by an appropriate moments condition. Another important