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A canonical setting and separating times for continuous local martingales

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Abstract

The notion of a *separating time* for a pair of measures on a filtered space is helpful for studying problems of (local) absolute continuity and singularity of measures. In this paper, we describe a certain canonical setting for continuous local martingales (abbreviated below as CLMs) and find an explicit form of separating times for CLMs in this setting.

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1. Introduction

1.1

The notion of a *separating time* for a pair of measures on a filtered space was introduced in [1] and applied there to study problems of (local) absolute continuity and singularity of measures (see also [2]). The definition and basic properties of separating times are recalled in Section 2.

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In Section 3, we describe a certain canonical setting for continuous local martingales (henceforth abbreviated as CLMs), which was introduced in [14]. By the Dambis–Dubins–Schwarz theorem, each CLM is a time-changed Brownian motion. An element of the canonical space for CLMs will be the pair consisting of a Brownian trajectory and a trajectory of a time-change. This amounts to saying that a CLM can be constructed by an appropriate probability kernel from the Wiener space into the time-change path space. Thus, the canonical setting can be viewed as converse to the Dambis–Dubins–Schwarz theorem.

In Section 4, we find an explicit form of separating times for CLMs in this canonical setting. In particular, we consider the special case of pure continuous local martingales (abbreviated as PCLMs). Pure continuous local martingales form an important subclass of CLMs and play a major role in the theory of stochastic processes (cf. Revuz and Yor [13]). Therefore, the Appendix contains some discussion of PCLMs difficult to access in the literature, and their relation to the canonical setting.

1.2

We adopt the usual convention $\inf \emptyset = \infty$. We say that the process M is a CLM if it is a CLM with respect to its natural filtration $\mathcal{F}_t^{0,M} = \sigma(M_s;\ s \in [0,t])$, $t \geq 0$.

Subsequently, we use the following two well-known facts without further explanation:

(i) If M is a CLM and $N \stackrel{\text{law}}{=} M$, then N is a CLM.
(ii) Let $(\mathcal{F}_t)$ be an arbitrary filtration. If M is an $(\mathcal{F}_t)$-CLM, then it is a CLM and an $(\mathcal{F}_t^M)$-CLM, where $\mathcal{F}_t^M = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^{0,M}$, $t \geq 0$.

2. Separating times

All results introduced in this section are taken from [1] (see also [2]).

Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$. We recall that the σ-field $\mathcal{F}_\tau$, for any $(\mathcal{F}_t)$-stopping time τ, is defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}\colon A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for any } t \in [0,\infty)\}.$$

In particular, $\mathcal{F}_\infty = \mathcal{F}$. Note that we do not assume here that $\mathcal{F} = \bigvee_{t\in[0,\infty)} \mathcal{F}_t$.

Let P and $\widetilde{\mathsf{P}}$ be probability measures on $\mathcal{F}$. As usual, P_τ (resp., $\widetilde{\mathsf{P}}_\tau$) denotes the restriction of P (resp., $\widetilde{\mathsf{P}}$) to $\mathcal{F}_\tau$.

In what follows, it will be convenient for us to consider the extended positive half-line $[0,\infty] \cup \{\delta\}$, where δ is an additional point. We order $[0,\infty] \cup \{\delta\}$ in the following way: we take the usual order on $[0,\infty]$ and let $\infty < \delta$.

Definition 2.1. An *extended stopping time* is a map $\tau\colon \Omega \to [0,\infty] \cup \{\delta\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for any $t \in [0,\infty]$.

In order to introduce the notion of a separating time, we need to formulate the following result.

Proposition 2.2. (i) *There is an extended stopping time S such that, for any stopping time τ,*

$$\widetilde{\mathsf{P}}_\tau \sim \mathsf{P}_\tau \quad \text{on the set } \{\tau < S\}, \tag{1}$$

$$\widetilde{\mathsf{P}}_\tau \perp \mathsf{P}_\tau \quad \text{on the set } \{\tau \geq S\}. \tag{2}$$

(ii) *If S' is another extended stopping time with these properties, then $S' = S$ $\mathsf{P}, \widetilde{\mathsf{P}}$-a.s.*