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Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds[☆]

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Abstract

A gradient-entropy inequality is established for elliptic diffusion semigroups on arbitrary complete Riemannian manifolds. As applications, a global Harnack inequality with power and a heat kernel estimate are derived.

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1. The main result

Let M be a non-compact complete connected Riemannian manifold, and P_t be the Dirichlet diffusion semigroup generated by $L = \Delta + \nabla V$ for some C^2 function V. We intend to establish reasonable gradient estimates and Harnack type inequalities for P_t. In case that $\mathrm{Ric} - \mathrm{Hess}_V$ is bounded below, a dimension-free Harnack inequality was established in [14] which, according

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to [15], is indeed equivalent to the corresponding curvature condition. See e.g. [2] for equivalent statements on heat kernel functional inequalities; see also [8,3,7] for a parabolic Harnack inequality using the dimension–curvature condition by shifting time, which goes back to the classical local parabolic Harnack inequality of Moser [9].

Recently, some sharp gradient estimates have been derived in [11,18] for the Dirichlet semigroup on relatively compact domains. More precisely, for $V = 0$ and a relatively compact open C^2 domain D, the Dirichlet heat semigroup P_t^D satisfies

$$|\nabla P_t^D f|(x) \le C(x,t) P_t^D f(x), \quad x \in D,\ t > 0, \tag{1.1}$$

for some locally bounded function $C\colon D\times]0,\infty[\to]0,\infty[$ and all $f \in \mathscr{B}_b^+$, the space of bounded non-negative measurable functions on M. Obviously, this implies the Harnack inequality

$$P_t^D f(x) \le \tilde{C}(x,y,t) P_t^D f(y), \quad t > 0,\ x,y \in D,\ f \in \mathscr{B}_b^+, \tag{1.2}$$

for some function $\tilde{C}\colon M^2\times]0,\infty[\to]0,\infty[$. The purpose of this paper is to establish inequalities analogous to (1.1) and (1.2) globally on the whole manifold M.

On the other hand however, both (1.1) and (1.2) are, in general, wrong for P_t in place of P_t^D. A simple counter-example is already the standard heat semigroup on $\mathbb{R}^d$. Hence, we turn to search for the following slightly weaker version of gradient estimate:

$$\begin{aligned}|\nabla P_t f(x)| &\le \delta \left[P_t(f \log f) - P_t f \log P_t f\right](x) + \frac{C(\delta,x)}{t\wedge 1} P_t f(x),\\ &x \in M,\ t > 0,\ \delta > 0,\ f \in \mathscr{B}_b^+,\end{aligned} \tag{1.3}$$

for some positive function $C\colon]0,\infty[\times M \to]0,\infty[$. When $\mathrm{Ric} - \mathrm{Hess}_V$ is bounded below, this kind of gradient estimate follows from [2, Proposition 2.6] but is new without curvature conditions. In particular, it implies the Harnack inequality with power introduced in [14] (see Theorem 1.2).

Theorem 1.1. *There exists a continuous positive function F on $]0,1]\times M$ such that*

$$\begin{aligned}|\nabla P_t f(x)| \le\ & \delta\,(P_t f \log f - P_t f \log P_t f)(x)\\ &+ \left(F(\delta\wedge 1, x)\left(\frac{1}{\delta(t\wedge 1)} + 1\right) + \frac{2\delta}{\mathrm{e}}\right) P_t f(x),\\ &\delta > 0,\ x \in M,\ t > 0,\ f \in \mathscr{B}_b^+.\end{aligned} \tag{1.4}$$

Theorem 1.2. *There exists a positive function $C \in C(]1,\infty[\times M^2)$ such that*

$$\begin{aligned}(P_t f(x))^\alpha &\le (P_t f^\alpha(y)) \exp\left\{\frac{2(\alpha-1)}{\mathrm{e}} + \alpha C(\alpha,x,y)\left(\frac{\alpha\rho^2(x,y)}{(\alpha-1)(t\wedge 1)} + \rho(x,y)\right)\right\},\\ &\alpha > 1,\ t > 0,\ x,y \in M,\ f \in \mathscr{B}_b^+,\end{aligned}$$

where ρ is the Riemannian distance on M. Consequently, for any $\delta > 2$ there exists a positive function $C_\delta \in C([0,\infty[\times M)$ such that the transition density $p_t(x,y)$ of P_t with respect to $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$, where $\mathrm{d}x$ is the volume measure, satisfies

$$p_t(x,y) \le \frac{\exp\left\{-\rho(x,y)^2/(2\delta t) + C_\delta(t,x) + C_\delta(t,y)\right\}}{\sqrt{\mu(B(x,\sqrt{2t}))\mu(B(y,\sqrt{2t}))}}, \quad x,y \in M,\ t \in]0,1[.$$