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Stochastic Processes and their Applications 119 (2009) 391–409
stochastic processes and their applications
www.elsevier.com/locate/spa

# Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion

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Received 1 March 2005; received in revised form 25 October 2006; accepted 4 February 2008
Available online 1 April 2008

**Abstract**

We prove the Malliavin regularity of the solution of a stochastic differential equation driven by a fractional Brownian motion of Hurst parameter $H > 0.5$. The result is based on the Fréchet differentiability with respect to the input function for deterministic differential equations driven by Hölder continuous functions. It is also shown that the law of the solution has a density with respect to the Lebesgue measure, under a suitable nondegeneracy condition.


*Keywords:* Stochastic differential equation; Malliavin calculus; Fractional Brownian motion

## 1. Introduction

Let $B = \{B_t, t \geq 0\}$ be an $m$-dimensional fractional Brownian motion (fBm for short) of Hurst parameter $H \in (0, 1)$. That is, $B$ is a centered Gaussian process with the covariance function $\mathbb{E}(B_s^i B_t^j) = R_H(s,t)\delta_{ij}$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1}$$

If $H = \frac{1}{2}$, $B$ is a Brownian motion. From (1), it follows that $\mathbb{E}\left(|B_t - B_s|^2\right) = m|t-s|^H$ so the process $B$ has $\alpha$-Hölder continuous paths for all $\alpha \in (0, H)$. We refer the reader to [13] and

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0304-4149/$ - see front matter 
doi:10.1016/j.spa.2008.02.016

references therein for further information about fBm and stochastic integration with respect to this process.

In this article we fix $\frac{1}{2} < H < 1$ and we consider the solution $\{X_t, t \in [0, T]\}$ of the following stochastic differential equation on $\mathbb{R}^d$:

$$X_t^i = x_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{ij}(X_s) \mathrm{d}B_s^j + \int_0^t b^i(X_s) \mathrm{d}s, \quad t \in [0, T], \tag{2}$$

$i = 1, \ldots, d$, where $x_0 \in \mathbb{R}^d$ is the initial value of the process $X$.

The stochastic integral in (2) is a pathwise Riemann–Stieltjes integral (see Young [16]). Suppose that $\sigma$ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{H} - 1$, and $b$ is Lipschitz; then there is a unique solution to Eq. (2) which has Hölder continuous trajectories of order $H - \varepsilon$, for any $\varepsilon > 0$. This result has been proved by Lyons in [8] in the case $b = 0$, using the $p$-variation norm. The theory of rough paths analysis introduced by Lyons in [9] was used by Coutin and Qian to prove an existence and uniqueness result for Eq. (2) in the case $H \in (\frac{1}{4}, \frac{1}{2})$ (see [4]).

The Riemann–Stieltjes integral appearing in Eq. (2) can be expressed as a Lebesgue integral using a fractional integration by parts formula (see Zähle [17]). Using this formula for the Riemann–Stieltjes integral, Nualart and Răşcanu have established in [14] the existence of a unique solution for a class of general differential equations that includes (2).

The main purpose of our work is to study the regularity of the solution to Eq. (2) in the sense of Malliavin calculus, and to show the absolute continuity for the law of $X_t$ for $t > 0$, assuming an ellipticity condition on the coefficient $\sigma$. First we establish a general result on the regularity with respect to the driven function for the solution of deterministic equations, using the techniques of fractional calculus developed in [14]. This allows us to deduce the differentiability of the solution to Eq. (2) in the direction of the Cameron–Martin space. These results are related to those proved by Lyons and Dong Li in [10] on the smoothness of Itô maps for such equations in terms of Fréchet–Gâteaux differentiability.

The regularity results obtained here have been used in a recent paper by Baudoin and Hairer [1] to show the smoothness of the density under a hypoellipticity Hörmander condition. This result requires also the existence of moments for the iterated derivatives, which has been established in [6]. In [11], the existence of a density for the solution of a one-dimensional equation is shown. See also [2,3] for other recent results.

The paper is organized as follows. In Section 2 we establish the Fréchet differentiability with respect to the input function for deterministic differential equations driven by Hölder continuous functions. Section 3 is devoted to analyzing stochastic differential equations driven by a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, the main result being the differentiability of the solution in the directions of the Cameron–Martin space. In Section 4 we prove the absolute continuity of the solution under ellipticity assumptions. The proofs of some technical results are given in the Appendix.

## 2. Deterministic differential equations driven by rough functions

We first introduce some preliminaries. Given a measurable function $f : [0, T] \to \mathbb{R}^d$ and $\alpha \in (0, \frac{1}{2})$, we will make use of the notation

$$\Delta_t^{\alpha}(f) = |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{|t - s|^{\alpha+1}} \mathrm{d}s.$$