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Multi-scaling of moments in stochastic volatility models

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Abstract

We introduce a class of stochastic volatility models $(X_t)_{t \geq 0}$ for which the absolute moments of the increments exhibit anomalous scaling: $\mathbb{E}\left(| X_{t+h} - X_t |^q\right)$ scales as $h^{q/2}$ for $q < q^*$, but as $h^{A(q)}$ with $A(q) < q/2$ for $q > q^*$, for some threshold q^*. This multi-scaling phenomenon is observed in time series of financial assets. If the dynamics of the volatility is given by a mean-reverting equation driven by a Levy subordinator and the characteristic measure of the Levy process has power law tails, then multi-scaling occurs if and only if the mean reversion is superlinear.

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1. Introduction

The last few decades have seen a considerable effort in constructing stochastic dynamics which exhibit some of the peculiar features of many observed time series, such as: heavy tailed distribution, long memory and path discontinuities. In particular, applications to mathematical finance have motivated the use of stochastic differential equations driven by general Levy processes. In this paper we consider a different, though related, pattern which is rather systematically

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observed in time series of financial assets, that we call *multi-scaling of moments* [21,15,14,12,11]. Let $(X_t)_{t\geq 0}$ be a continuous-time martingale, having stationary increments; in financial applications this could be identified with the de-trended *log-price* of an asset, or the price with respect to the martingale measure used to price derivatives. We say the multi-scaling of moments occurs if the limit

$$\limsup_{h\downarrow 0} \frac{\log \mathbb{E}\left(|X_{t+h}-X_t|^q\right)}{\log h} =: A(q) \tag{1.1}$$

is *non-linear* on the set $\{q \geq 1 : |A(q)| < +\infty\}$. More intuitively, (1.1) says that $\mathbb{E}\left(|X_{t+h}-X_t|^q\right)$ scales, in the limit as $h \downarrow 0$, as $h^{A(q)}$, with $A(q)$ non-linear. In the case X_t is a Brownian martingale (i.e. a stochastic integral w.r.t. a Brownian motion), one would expect $A(q) = \frac{q}{2}$, at least for q sufficiently small. In this case, multi-scaling of moments can be identified with deviations from this *diffusive* scaling, occurring for q above a given threshold; this type of multi-scaling is indeed observed in the values of many financial indices and exchange rates.

A class of stochastic processes that exhibit multi-scaling for a rather arbitrary scaling function $A(q)$ are the so-called *multifractal models* [5,4,6]. In these models, the process X_t is given as the *random time change* of a Brownian motion:

$$X_t := W_{I(t)}, \tag{1.2}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, and $I(t)$ is a stochastic process, often taken to be independent of $W_\cdot$, with continuous and increasing trajectories, sometimes called *trading time*. Modeling financial series through a random time-change of Brownian motion is a classical topic, dating back to Clark [9], and reflects the natural idea that external information influences the speed at which exchanges take place in a market. In multifractal models, the trading time $I(t)$ is a process with *non absolutely continuous* trajectories. As a consequence, X_t cannot be written as a *stochastic volatility model*, i.e. in the form $dX_t = \sigma_t dB_t$, for a Brownian motion B_t. This makes the analysis of multifractal models hard in many respects, as the standard tools of Ito's Calculus cannot be applied.

In [1] a much simpler process has been constructed which exhibits a bi-scaling behavior: (1.1) hold with a function $A(q)$ which is piecewise linear and the slope $A'(q)$ takes two different values, which suffices to fit most of the cases observed. This process is a stochastic volatility model, although of a rather peculiar type. Besides exhibiting multi-scaling, this model accounts for other relevant *stylized facts* in time series of financial indices, such as the *autocorrelation* profile $t \mapsto Cov(|X_h - X_0|, |X_{t+h} - X_t|)$ as well as heavy tailed distribution of $X_h - X_0$.

The aim of this paper is to analyze multi-scaling in a more general class of stochastic volatility models, namely those of the form $dX_t = \sigma_t dB_t$, with a volatility process σ_t independent of the Brownian motion B_t; these processes are exactly those that can be written in the form (1.2) with a trading time $I(t)$ independent of W_t, and with *absolutely continuous* trajectories. We devote special attention to models in which $V_t := \sigma_t^2$ is a *stationary* solution of a stochastic differential equation of the form

$$dV_t = -f(V_t)dt + dL_t, \tag{1.3}$$

for a *Levy subordinator* L_t whose characteristic measure has *power law tails at infinity*, and a function $f(\cdot)$ such that a stationary solution exists, and it is unique in law. We first show multi-scaling is *not* possible if $f(\cdot)$ has *linear growth*. Thus, the heavy tails produced by the Levy process are not sufficient to produce multi-scaling. On the other hand, we show that, if $f(\cdot)$