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Infinite-dimensional stochastic differential equations related to Bessel random point fields

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Abstract

We solve the infinite-dimensional stochastic differential equations (ISDEs) describing an infinite number of Brownian particles in $\mathbb{R}^+$ interacting through the two-dimensional Coulomb potential. The equilibrium states of the associated unlabeled stochastic dynamics are Bessel random point fields. To solve these ISDEs, we calculate the logarithmic derivatives, and prove that the random point fields are quasi-Gibbsian.

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1. Introduction

The Bessel random point fields μ_α $(-1 < \alpha < \infty)$ are probability measures on the configuration space S over $S = [0, \infty)$, whose n-point correlation functions ρ_α^n (see (2.2)) with respect to the Lebesgue measure are given by

$$\rho_\alpha^n(x_1, \dots, x_n) = \det[\mathsf{K}_\alpha(x_i, x_j)]_{1 \le i, j \le n}. \tag{1.1}$$

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Here, $\mathsf{K}_\alpha(x, y)$ is a continuous function called the Bessel kernel defined with the Bessel function J_α of order α such that for $x \neq y$

$$\begin{aligned}\mathsf{K}_\alpha(x, y) &= \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y})}{2(x-y)} \\ &= \frac{\sqrt{x}J_{\alpha+1}(\sqrt{x})J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{x})\sqrt{y}J_{\alpha+1}(\sqrt{y})}{2(x-y)}\end{aligned} \tag{1.2}$$

and that for $x = y$

$$\mathsf{K}_\alpha(x, x) = \frac{1}{4}\{J_\alpha(\sqrt{x})^2 - J_{\alpha+1}(\sqrt{x})J_{\alpha-1}(\sqrt{x})\}. \tag{1.3}$$

Note that $0 \leq \mathsf{K}_\alpha \leq \mathrm{Id}$ as an operator on $L^2(S, dx)$. By definition μ_α are determinantal random point fields with Bessel kernels K_α (see [26]).

It is known that these random point fields arise as a scaling limit at the hard left edge of the distributions μ_α^{n} of the spectrum of the Laguerre ensemble. The random point fields μ_α represent the thermodynamic limit of the n-particle systems μ_α^{n}, whose labeled densities $\sigma_\alpha^{\mathsf{n}}(\mathbf{x})d\mathbf{x}$ are given by

$$\sigma_\alpha^{\mathsf{n}}(\mathbf{x}) = \frac{1}{\mathcal{Z}_\alpha^{\mathsf{n}}} e^{-\sum_{i=1}^{\mathsf{n}} x_i/4\mathsf{n}} \prod_{j=1}^{\mathsf{n}} x_j^\alpha \prod_{k<l}^{\mathsf{n}} |x_k - x_l|^2. \tag{1.4}$$

Very loosely, by taking n to infinity, we obtain the following informal expression for the μ_α:

$$\mu_\alpha(d\mathbf{x}) = \frac{1}{\mathcal{Z}_\alpha^\infty} \prod_{j=1}^{\infty} x_j^\alpha \prod_{k<l}^{\infty} |x_k - x_l|^2 \prod_{m=1}^{\infty} dx_m. \tag{1.5}$$

Hence we regard the μ_α as random point fields with free potentials $\Phi_\alpha(x) = -\alpha \log x$ and interaction potential $\Psi(x) = -2\log|x|$. Unlike Ruelle's class of interaction potentials, one cannot justify this using the Dobrushin–Lanford–Ruelle (DLR) equations. Instead, we will proceed in terms of logarithmic derivatives in Theorem 2.3.

We next turn to the stochastic dynamics associated with the μ_α^{n}. To prevent the particles from hitting the origin, we suppose that $1 \leq \alpha$ (Lemma B.1). Then, from Eq. (1.4), it can be seen that the natural n-particle stochastic dynamics $\mathbf{X}^{\mathsf{n}} = (X_t^{\mathsf{n},1}, \ldots, X_t^{\mathsf{n},\mathsf{n}})$ are given by the stochastic differential equations (SDEs)

$$dX_t^{\mathsf{n},i} = dB_t^i + \left\{-\frac{1}{8\mathsf{n}} + \frac{\alpha}{2X_t^{\mathsf{n},i}} + \sum_{j\neq i}^{\mathsf{n}} \frac{1}{X_t^{\mathsf{n},i} - X_t^{\mathsf{n},j}}\right\} dt \quad (1 \leq i \leq \mathsf{n}). \tag{1.6}$$

Hence, taking n to infinity, we come to the ISDEs

$$dX_t^i = dB_t^i + \left\{\frac{\alpha}{2X_t^i} + \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\right\} dt \quad (i \in \mathbb{N}). \tag{1.7}$$

The purpose of this paper is to solve these ISDEs in such a way that the equilibrium states of the associated unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^\infty \delta_{X_t^i}$ are Bessel random point fields μ_α.