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The alternating marked point process of h-slopes of drifted Brownian motion

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Abstract

We show that the slopes between h-extrema of the drifted 1D Brownian motion form a stationary alternating marked point process, extending the result of J. Neveu and J. Pitman for the non-drifted case. Our analysis covers the results on the statistics of h-extrema obtained by P. Le Doussal, C. Monthus and D. Fisher via a Renormalization Group analysis and gives a complete description of the slope between h-extrema covering the origin by means of the Palm–Khinchin theory. Moreover, we analyze the behavior of the Brownian motion near its h-extrema.

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Keywords: Brownian motion; Marked point processes; Palm–Khinchin theory; Fluctuation theory

1. Introduction

Let B be a two-sided standard Brownian motion with drift $-\mu$. Given $h > 0$ we say that B admits an *h-minimum* at $x \in \mathbb{R}$, and that x is a point of h-minimum, if there exist $u < x < v$ such that $B_t \geq B_x$ for all $t \in [u, v]$, $B_u \geq B_x + h$ and $B_v \geq B_x + h$. Similarly, we say that B admits an *h-maximum* at $x \in \mathbb{R}$, and that x is a point of h-maximum, if there exist $u < x < v$ such that $B_t \leq B_x$ for all $t \in [u, v]$, $B_u \leq B_x - h$ and $B_v \leq B_x - h$. We say that B admits an *h-extremum* at $x \in \mathbb{R}$, and that x is a point of h-extremum, if x is a point of h-minimum or a point of h-maximum. Finally, the truncated trajectory B going from an h-minimum to an h-maximum

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will be called upward h-slope, while the truncated trajectory B going from an h-maximum to an h-minimum will be called downward h-slope.

Our first object of investigation is the statistics of h-slopes. The non-drifted case $\mu = 0$ has been studied in [1]. Here we assume $\mu \neq 0$ and show (see Theorem 1) that the statistics of h-slopes is well described by a stationary alternating marked simple point process on $\mathbb{R}$ whose points are the points of h-extrema of the Brownian motion, and each point x is marked by the h-slope going from x to the subsequent point of h-extremum. We will show that the h-slopes are independent and specify the laws P_+^μ, P_-^μ of upward h-slopes and downward h-slopes not covering the origin, respectively. The h-slope covering the origin shows a different distribution that can be derived by means of the Palm–Khinchin theory [2,3].

Our proof is based both on fluctuation theory for Lévy processes, and on the theory of marked simple point processes. The part of fluctuation theory follows strictly the scheme of [1] and can be generalized to spectrally one-sided Lévy processes, i.e. real valued random processes with stationary independent increments and with no positive jumps or with no negative jumps [4] [Chapter VII]. In fact, some of the identities of Lemma 1 and Proposition 1 below have already been obtained with more sophisticated methods for general spectrally one-sided Lévy processes (see [5–7] and references therein). On the other hand, the description of the h-slopes as a stationary alternating marked simple point process allows using the very powerful Palm–Khinchin theory, which extends renewal theory and leads to a complete description of the h-slope covering the origin. This analysis can be easily extended to more general Lévy processes, as the ones treated in [7].

As discussed in Section 3, our results concerning the statistics of h-extrema of drifted Brownian motion correspond to the ones obtained in [8] via a non-rigorous Real Space Renormalization Group method applied to Sinai random walk with a vanishing bias. In addition to a rigorous derivation, here we are able to describe also the statistics of the h-slopes, lacking in [8].

In Section 5 (see Theorem 2), we analyze the behavior of the drifted Brownian motion around its h-extrema. While in the non-drifted case a generic h-slope not covering the origin behaves in proximity of its extremes as a 3D Bessel process, in the drifted case it behaves as a process with a cothangent drift, satisfying the SDE

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}\beta_t \pm \mu \coth(\mu X_t)\,\mathrm{d}t, & t \geq 0, \\ X_0 = 0, \end{cases} \tag{1.1}$$

where β_t is a standard Brownian motion and the sign in the r.h.s. depends on the kind of h-slope (downward or upward) and on the kind of h-extrema (h-minimum or h-maximum). In addition, we show that the process (1.1) is simply the Brownian motion on $[0, \infty)$, starting at the origin, with drift $\pm\mu$, Doob-conditioned to hit $+\infty$ before 0.

The interest in the statistics of h-slopes and their behavior near the extremes comes also from the fact that, considering the diffusion in a drifted Brownian potential, the piecewise linear path obtained by connecting the h-extrema of the Brownian potential is the effective potential for the diffusion at large times [9].

2. Statistics of h-slopes of drifted Brownian motion

Given $\mu, x \in \mathbb{R}$ we denote by $\mathbf{P}_x^\mu$ the law on $C(\mathbb{R}, \mathbb{R})$ of the standard two-sided Brownian motion B with drift $-\mu$ having value x at time zero, i.e. $B_t = x + B_t^* - \mu t$ where $B^* : \mathbb{R} \to \mathbb{R}$