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Continuity in the Hurst index of the local times of anisotropic Gaussian random fields

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Abstract

Let $\left\{\{X^H(t), t \in \mathbb{R}^N\}, H \in (0,1)^N\right\}$ be a family of (N, d)-anisotropic Gaussian random fields with generalized Hurst indices $H = (H_1, \ldots, H_N) \in (0,1)^N$. Under certain general conditions, we prove that the local time of $\{X^{H^0}(t), t \in \mathbb{R}^N\}$ is jointly continuous whenever $\sum_{\ell=1}^N 1/H_\ell^0 > d$. Moreover we show that, when H approaches H^0, the law of the local times of $X^H(t)$ converges weakly [in the space of continuous functions] to that of the local time of X^{H^0}. The latter theorem generalizes the result of [M. Jolis, N. Viles, Continuity in law with respect to the Hurst parameter of the local time of the fractional Brownian motion, J. Theoret. Probab. 20 (2007) 133–152] for one-parameter fractional Brownian motions with values in $\mathbb{R}$ to a wide class of (N, d)-Gaussian random fields. The main argument of this paper relies on the recently developed sectorial local nondeterminism for anisotropic Gaussian random fields.

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1. Introduction

Gaussian random fields have been extensively studied in probability theory and applied in many scientific areas including physics, engineering, hydrology, biology, economics, just

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to mention a few. Since many data sets from various areas such as image processing, hydrology, geostatistics and spatial statistics have anisotropic nature in the sense that they have different geometric and probabilistic characteristics along different directions, many authors have proposed applying anisotropic Gaussian random fields as more realistic models. See, for example, [1–4].

Several classes of anisotropic Gaussian random fields have been introduced and studied for theoretical and application purposes. For example, Kamont [5] introduced fractional Brownian sheets and studied some of their regularity properties. Benassi et al. [6] and Bonami and Estrade [3] considered some anisotropic Gaussian random fields with stationary increments. Anisotropic Gaussian random fields also arise naturally in stochastic partial differential equations (see, e.g., [7–10]), in studying the most visited sites of symmetric Markov processes [11], and as spatial or spatiotemporal models in statistics (e.g., [2,12,13]).

Many of these anisotropic Gaussian random fields are governed by their generalized Hurst indices $H \in (0, 1)^N$ (see Section 2 for the definition of a generalized Hurst index). People often have to use a statistical estimate of the index H in practice since the exact value of the index is unknown in general. Therefore, a justification of the use of a model is needed in application with an unknown H. Motivated by this purpose, Jolis and Viles [14] investigated the continuity in law with respect to the Hurst parameter of the local time of real-valued fractional Brownian motions. They proved that the law of the local times of the fractional Brownian motions with Hurst index α converges weakly to that of the local time of fractional Brownian motion with Hurst index α_0, when α tends to α_0. However, the method they developed there depends heavily on the one-parameter setting and the explicit covariance structure of fractional Brownian motion. It seems hard to apply the method of Jolis and Viles [14] to Gaussian random fields, where "time" parameters are vectors and their covariance structures are more complicated in general.

The main objective of this paper is to provide a general method for studying the continuity of the laws of the local times of Gaussian random fields. More precisely, we prove that, under some mild conditions, the law (in the space of continuous functions) of the local times of (N, d)-anisotropic Gaussian random fields with generalized Hurst indices H converges weakly to that of the local time of an (N, d)-anisotropic Gaussian field with index H^0, when H approaches H^0. Our result generalizes the result of Jolis and Viles [14] for real-valued fractional Brownian motion to a wide class of (N, d)-anisotropic Gaussian random fields, including fractional Brownian sheets, anisotropic Gaussian fields with stationary increments and the spatiotemporal models in [12,13]. The main ingredient we use in our proof is the recently developed properties of sectorial local nondeterminism for anisotropic Gaussian random fields, see [15–17].

The rest of this paper is organized as follows. Section 2 states the general condition (i.e., Condition A) on Gaussian random fields under investigation. We show that these conditions are satisfied by several classes of Gaussian random fields which are of importance in theory and/or in applications. In Section 3, we recall the definition of local times of vector fields and prove the existence and joint continuity of the local times of Gaussian random fields satisfying Condition A. The key estimate for this paper is stated as Lemma 3.2. In Section 4, we prove the tightness of the laws of local time $\{L^H\}$ as H belongs to a neighborhood of a fixed index $H^0 \in (0, 1)^N$. In Section 5, we study the convergence in law of local times of the family of Gaussian random fields satisfying Condition A. Finally, we give the proof of our key lemma, Lemma 3.2, in Section 6.

Throughout this paper, we use $\langle \cdot, \cdot \rangle$ and $| \cdot |$ to denote the ordinary scalar product and the Euclidean norm in $\mathbb{R}^m$ respectively, no matter what the value of the integer m is. In Section i, unspecified positive and finite constants will be numbered as $c_{i,1}, c_{i,2}, \ldots$.