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Zonal polynomials and a multidimensional quantum Bessel process

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Abstract

Using the machinery of zonal polynomials, we construct a Markov process which is a multidimensional analogue of Biane's quantum Bessel process.

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1. Introduction

Zonal polynomials, introduced by A. T. James [17], are of fundamental importance in the theory of special functions of matrix argument and for multivariate statistical analysis. In this paper, we shall show that zonal polynomials play a key role in the construction of an intriguing stochastic process (see Theorem 3.1), which is a multidimensional analogue of the quantum Bessel process introduced by Philippe Biane in his very interesting paper [4].

The adjective "quantum" (Biane's reason for using it is explained in Section 1.2) might be a little bit misleading, as the quantum Bessel process, as well as its multidimensional analogue

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constructed in Theorem 3.1 of this paper, are just ordinary (classical, commutative) Markov processes living on some subsets of $\mathbb{R}^d$. Although built upon some noncommutative objects, all processes we consider here are classical and can entirely be described using classical (commutative) probability theory.

We shall begin by sketching an approach to the construction of the usual Bessel process that will shed some light on what follows.

(To state the results that will follow requires introduction of a significant amount of machinery, including some basics of C*-algebras or Gelfand pairs and spherical functions. To make the paper accessible to a wide audience, we decided to define in the Appendix a fair amount of the notions and to present there some of the results that we use in the Introduction and Section 2, and the reader should refer to Appendix as necessary.)

1.1. A construction of Bessel process

Consider the additive group $G = \mathbb{R}^n$ and let $\psi(x) = -|x|^2/2$ with $x \in G$ and $|\cdot|$ being the usual Euclidean norm on $\mathbb{R}^n$. It is well known that for all $t \geq 0$ the function $x \mapsto \exp[t\psi(x)]$ is positive definite on G. Define a semigroup $(Q_t)_t$ on the L^1 group algebra of G via

$$Q_t f(g) = \exp[t\psi(g)] f(g) \tag{1.1}$$

(for $f \in L^1(G)$, $g \in G$ and $t \geq 0$). It is easy to observe that $(Q_t)_t$ is a semigroup of positive contractions on $L^1(G)$. This semigroup extends in a natural way to the semigroup of positive contractions on the group C*-algebra $C^*(G)$ of G.

Since G is abelian, $C^*(G)$ is commutative and therefore, by the Gelfand–Naimark theorem, it is naturally isomorphic to the algebra $C_0(\hat{G})$ of continuous functions on the dual group $\hat{G}$, vanishing at infinity. Since the dual group of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ we obtain that $(Q_t)_t$ is the convolution semigroup that convolves functions with the measures with the Fourier transforms $\exp(-t|x|^2/2)$. That is, $(Q_t)_t$ is the semigroup of Brownian motion on $\mathbb{R}^n$ (the heat semigroup on $\mathbb{R}^n$).

The orthogonal group $O(n)$ acts on G by automorphisms and the C*-subalgebra of radial elements of $C^*(G)$ ("radial" means invariant under the action of $O(n)$ or, equivalently, depending on $|\cdot|$ only) is isomorphic to $C_0(\mathbb{R}_+)$. It is well known, and can be shown by the methods similar to those used below in Appendix A.5 (modified appropriately to handle the case of the C*-subalgebra of abelian subgroup of G rather than C*-subalgebra associated with a Gelfand pair), that the restriction of the semigroup $(Q_t)_t$ to $C_0(\mathbb{R}_+)$ is given by the Bessel semigroup

$$p_t(x, \mathrm{d}y) = \frac{y^{n-1}}{(xy)^{n/2-1}t} I_{n/2-1}\left(\frac{xy}{t}\right) \exp\left(-\frac{x^2+y^2}{2t}\right) \mathbb{1}_{(0,\infty)}(y)\mathrm{d}y$$

(I_ν is the modified Bessel function of the first kind), whose corresponding Markov process is Bessel process (see e.g. [23]). This is not surprising since Bessel processes are known to be the radial parts of Brownian motion.

1.2. Biane's quantum Bessel process

Philippe Biane in his very interesting paper [4] introduces a classical Markov process, the quantum Bessel process, which in many aspects resembles the ordinary Bessel process. It is constructed in a way similar to the one described in the previous subsection.