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A moment problem for random discrete measures

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Abstract

Let X be a locally compact Polish space. A random measure on X is a probability measure on the space of all (nonnegative) Radon measures on X. Denote by $\mathbb{K}(X)$ the cone of all Radon measures η on X which are of the form $\eta = \sum_i s_i \delta_{x_i}$, where, for each i, $s_i > 0$ and δ_{x_i} is the Dirac measure at $x_i \in X$. A random discrete measure on X is a probability measure on $\mathbb{K}(X)$. The main result of the paper states a necessary and sufficient condition (conditional upon a mild *a priori* bound) when a random measure μ is also a random discrete measure. This condition is formulated solely in terms of moments of the random measure μ. Classical examples of random discrete measures are completely random measures and additive subordinators, however, the main result holds independently of any independence property. As a corollary, a characterization via moments is given when a random measure is a point process.

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1. Introduction

Let X be a locally compact Polish space, and let $\mathcal{B}(X)$ denote the associated Borel σ-algebra. For example, X can be the Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$. Let $\mathbb{M}(X)$ denote the space of all (non-

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negative) Radon measures on $(X, \mathcal{B}(X))$. The space $\mathbb{M}(X)$ is equipped with the vague topology. Let $\mathcal{B}(\mathbb{M}(X))$ denote the Borel σ-algebra on $\mathbb{M}(X)$.

Let us define the *cone of (nonnegative) discrete Radon measures on* X by

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) \,\middle|\, s_i > 0,\ x_i \in X \right\}.$$

Here δ_{x_i} denotes the Dirac measure with mass at x_i. In the above representation, the atoms x_i are assumed to be distinct, i.e., $x_i \neq x_j$ for $i \neq j$, and their total number is at most countable. By convention, the cone $\mathbb{K}(X)$ contains the null mass $\eta = 0$, which is represented by the sum over an empty set of indices i. As shown in [8], $\mathbb{K}(X) \in \mathcal{B}(\mathbb{M}(X))$.

A *random measure on* X is a measurable mapping $\xi : \Omega \to \mathbb{M}(X)$, where $(\Omega, \mathcal{F}, P)$ is a probability space, see e.g. [5,6,9]. A random measure which takes values in $\mathbb{K}(X)$ with probability one will be called a *random discrete measure*. We will give results which characterize when a random measure is a random discrete measure in terms of its moments.

Let us recall the classical characterization of a completely random measure by Kingman [11,6]. A random measure ξ is called *completely random* if, for any mutually disjoint sets $A_1, \dots, A_n \in \mathcal{B}(X)$, the random variables $\xi(A_1), \dots, \xi(A_n)$ are independent. Kingman's theorem states that every completely random measure ξ can be represented as $\xi = \xi_d + \xi_f + \xi_r$. Here ξ_d, ξ_f, ξ_r are independent completely random measures such that: ξ_d is a deterministic measure on X without atoms; ξ_f is a random measure with fixed (non-random) atoms, that is there exist a deterministic countable collection of points $\{x_i\}$ in X and non-negative independent random numbers $\{a_i\}$ with $\xi_f = \sum_i a_i \delta_{x_i}$; finally the most essential part ξ_r is an extended marked Poisson process which has no fixed atoms, in particular with probability one ξ_r is of the form $\sum_j b_j \delta_{y_j}$, where $\{b_j\}$ are non-negative random numbers and $\{y_j\}$ are random points in X.

Thus, by Kingman's result a completely random measure is a random discrete measure up to a non-random component. If one drops the assumption that the random measure is completely random, one cannot expect anymore to concretely characterize the distribution of ξ. Thus, a natural appropriate question is to ask when a random measure is a random discrete measure. One may be tempted to replace the assumption of complete randomness by a property of a sufficiently strong decay of correlation. However, the result of this paper shows that such an assumption cannot be sufficient.

Note that, in most interesting examples of completely random measures, the set of atoms of the random discrete measure is almost surely dense in X. A study of countable dense random subsets of X leads to "situations in which probabilistic statements about such sets can be uninformative" [10], see also [2]. It is the presence of the weights s_i in the definition of a random discrete measure that makes a real difference.

An important characteristic of a random measure is its moment sequence. We say that a random measure ξ has finite moments of all orders if, for each $n \in \mathbb{N}$ and all bounded subsets $A \in \mathcal{B}(X)$,

$$\mathbb{E}[\xi(A)^n] < \infty.$$

Then, the *n-th moment measure of* ξ is the unique symmetric measure $M^{(n)} \in \mathbb{M}(X^n)$ defined by the following relation

$$\forall A_1, \dots, A_n \in \mathcal{B}(X) : \quad M^{(n)}(A_1 \times \cdots \times A_n) := \mathbb{E}[\xi(A_1) \cdots \xi(A_n)].$$