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Bootstrap random walks

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Abstract

Consider a one dimensional simple random walk $X = (X_n)_{n\geq 0}$. We form a new simple symmetric random walk $Y = (Y_n)_{n\geq 0}$ by taking sums of products of the increments of X and study the two-dimensional walk $(X, Y) = ((X_n, Y_n))_{n\geq 0}$. We show that it is recurrent and when suitably normalised converges to a two-dimensional Brownian motion with independent components; this independence occurs despite the functional dependence between the pre-limit processes. The process of recycling increments in this way is repeated and a multi-dimensional analog of this limit theorem together with a transience result are obtained. The construction and results are extended to include the case where the increments take values in a finite set (not necessarily $\{-1, +1\}$).

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1. Introduction

Consider a symmetric simple random walk

$$X_n = \sum_{k=1}^{n} \xi_k, \quad n \geq 1, \text{ and } X_0 = 0,$$

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where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables with

$$\mathbb{P}(\xi_1 = -1) = \mathbb{P}(\xi_1 = +1) = \frac{1}{2}.$$

It is easy to see that the sequence

$$\eta_n = \prod_{k=1}^{n} \xi_k, \quad n \geq 1,$$

is made up of independent and identically distributed random variables taking values ± 1 with equal probability. It immediately follows that

$$Y_n = \sum_{k=1}^{n} \eta_k, \quad n \geq 1 \text{ and } Y_0 = 0$$

is also a symmetric simple random walk; that is

$$(Y_n)_{n\geq 0} \stackrel{d}{=} (X_n)_{n\geq 0}. \tag{1}$$

We refer to the process of constructing $(Y_n)_n$ from $(X_n)_n$ – that is of "recycling" the increments of the latter to form those of the former – as bootstrapping.

While (1) is immediately clear, what may be less understood is the behaviour of the two-dimensional process $W_n = (X_n, Y_n)$.

It is worth emphasising at this point in time that the filtrations generated by the two processes $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ are identical:

$$\eta_n = \prod_{k=1}^{n} \xi_k \quad \text{and} \quad \xi_n = \frac{\eta_n}{\eta_{n-1}} = \eta_{n-1}\eta_n.$$

This strong (functional) dependence is however entirely lost at infinity. More precisely, we establish that the process $(W_n)_{n\geq 0}$ suitably normalised converges (weakly) to a two-dimensional Brownian motion (with independent components). The process of taking partial products and their partial sums can then be iterated yielding a higher dimensional version of this result. Again, despite the functional dependence between the components of the pre-limit processes, the limiting process is a multidimensional Brownian motion (with independent components).

In this paper, we take a further generalising step, one that drops the requirement that $\xi_n \in \{-1, +1\}$. Instead, we allow ξ_n to take values in a finite set $\mathbb{U} = \{u_0, u_1, \ldots, u_{p-1}\} \subset \mathbb{R}$ and propose a general method for defining η_n and all other iterates in such way that all partial-sum processes are identical in distribution to $(X_n)_n$. Here again, the strong dependence in the joint process is lost at infinity and the limiting process is a multidimensional Brownian motion (with independent components). The functional central limit theorem in this generalised form is presented in Section 4.

We also briefly discuss a connection with cellular automata (see Section 3).

The pre-limit process W_n in itself is worth looking at and we present some of its properties in Section 2. Section 3 deals with the model setup and presents a number of basic properties including a rather precise formulation of the iterates, of any order. A number of combinatorial proofs are given in Section 5.