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Asymptotic formula for the tail of the maximum of smooth stationary Gaussian fields on non locally convex sets

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Abstract

In this paper we consider the distribution of the maximum of a Gaussian field defined on non locally convex sets. Adler and Taylor or Azaïs and Wschebor give the expansions in the locally convex case. The present paper generalizes their results to the non locally convex case by giving a full expansion in dimension 2 and some generalizations in higher dimension. For a given class of sets, a Steiner formula is established and the correspondence between this formula and the tail of the maximum is proved. The main tool is a recent result of Azaïs and Wschebor that shows that under some conditions the excursion set is close to a ball with a random radius. Examples are given in dimension 2 and higher.

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1. Introduction

Let $\mathcal{X} = \{X(t) : t \in S \subset \mathbb{R}^n\}$ be a random field with real values and let M_S be its maximum (or supremum) on S. Computing the distribution of the maximum is a very important issue from

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the theoretical point of view and also has a great impact on applications, especially in spatial statistics. This problem has therefore received a great deal of attention from many authors.

However an exact result is known only in very few cases, (see Azaïs and Wschebor [5]). In other cases, the only available results are asymptotic expansions or bounds mainly in the case of stationary Gaussian random fields.

One of the most well-known and quite general methods is the "double-sum method", first proposed by Pickands [13] and extended by Piterbarg [14,15]. The main idea of this method is to use the inclusion–exclusion principle and the Bonferroni inequality after dividing the parameter set into suitable smaller subsets. It was first proposed in dimension 1 : $n = 1$ (in this case we use the classical terminology of "random processes" instead of "random field"). More precisely, for some particular processes, i.e., the "α processes", Pickands proposed an equivalent for the tail of the maximum. However, the result depends on some unknown constants, referred to as Pickands' constants and just gives an equivalent.

Another method is the "tube method" proposed by Sun [18]. She observed that if the Karhunen–Loève expansion of the field is finite in the sense that there exist a finite number of random variables $\xi_1, \dots, \xi_k \overset{i.i.d}{\sim} \mathcal{N}(0, 1)$ such that at every point t in the parameter set, the value of the field at this point $X(t)$ can be expressed as

$$X(t) = a_t^1 \xi_1 + \cdots + a_t^k \xi_k,$$

where the vector $(a_t^1, \dots, a_t^k)$ has unit norm since $\mathrm{Var}(X(t)) = 1$, then the original parameter set can be transformed into a subset of the unit sphere $\mathcal{S}^{k-1}$. She then used Weyl's tube formula to compute the polynomial expansion of the volume of the tube around a subset of the unit sphere and derived the asymptotic formula of the tail of the distribution of the maximum from this expansion. She is the first one who realizes the strong connection between the geometric functionals of the parameter set (the coefficients of the polynomial expansion) and the tail of the distribution. When the Karhunen–Loève expansion is not finite, she uses a truncation argument to derive an asymptotic formula with two terms. Later on, this method was extended by Takemura and Kuriki [19,20].

In the 1940s, in his pioneering work, Rice [16] considered a stationary process $\mathcal{X}$ with $\mathcal{C}^1$ paths defined on the compact interval $[0, T]$. He observed that for every level u:

$$\begin{aligned}\mathbb{P}\left(\max_{t\in[0,T]} X(t) \geq u\right) &\leq \mathbb{P}(X(0) \geq u) + \mathbb{P}\left(\exists t \in [0, T] :\ X(t) = u, X'(t) \geq 0\right) \\ &\leq \mathbb{P}(X(0) \geq u) + \mathbb{E}\left(\mathrm{card}\left\{t \in [0, T] : X(t) = u, X'(t) \geq 0\right\}\right),\end{aligned}$$

where the last expectation can be evaluated by the famous Rice–Kac formula. This upper bound was later proved to be sharp by Piterbarg [17]. This Rice–Kac formula is the starting point of the following methods dealing with the random fields: the "Rice method" by Azaïs and Delmas [3,8], the "direct method" by Azaïs and Wschebor [5] and the "Euler characteristic method" by Adler and Taylor [1]. These methods use a multidimensional Rice–Kac formula: Generalized Rice formula (Azaïs and Wschebor) or Metatheorem (Adler and Taylor).

In the direct method, Azaïs and Wschebor used some results from the random matrix theory to compute the expectation of the absolute value of the determinant of the Hessian that appears in the Rice formula. They obtained an upper bound for the tail of the distribution depending on some geometric functionals of the parameter set. This upper bound is also sharp.

Adler and Taylor combined differential and integral geometry to find the "Euler characteristic method" that gives one of most frequently used results in this area. They considered stratified