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Frequently visited sites of the inner boundary of simple random walk range

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Abstract

This paper considers the question: how many times does a simple random walk revisit the most frequently visited site among the inner boundary points? It is known that in $\mathbb{Z}^2$, the number of visits to the most frequently visited site among all of the points of the random walk range up to time n is asymptotic to $\pi^{-1}(\log n)^2$, while in $\mathbb{Z}^d (d \geq 3)$, it is of order $\log n$. We prove that the corresponding number for the inner boundary is asymptotic to $\beta_d \log n$ for any $d \geq 2$, where β_d is a certain constant having a simple probabilistic expression.

Keywords: Inner boundary; Random walk range; Frequently visited site

1. Introduction

Many works have studied properties of the trajectory of a simple random walk. These properties include the growth rate of the trajectory's range, location of the most frequently visited site, the number of the visits to this site, the number of the sites of frequent visits, and so forth. There remain many interesting unsolved questions concerning these properties. The most frequently visited site among all the points of the range (of the walk of finite length) is called a favorite site and a site which is revisited many times (in a certain specified sense) is called a frequently visited site. About fifty years ago, Erdős and Taylor [5] posed a problem concerning a simple random walk in $\mathbb{Z}^d$: how many times does the random walk revisit the favorite site (up to

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a specific time)? Open problems concerning the favorite site are raised by Erdős and Révész [3,4] and Shi and Tóth [12] but remain unsolved so far. By Lifshits and Shi [9] it is known that the favorite site of a 1-dimensional random walk tends to be far from the origin, but almost nothing is known about its location for multi-dimensional walks.

In this paper, we focus on the most frequently visited site among the inner boundary points of the random walk range, rather than among all of the points of the range, and propose the question: how many times does a random walk revisit the most frequently visited site among the inner boundary points? Here, we briefly state our result and compare it with known results for the favorite site. Let $M_0(n)$ be the number of visits to the favorite site by the walk up to time n and $M(n)$ be that of the most frequently visited site among the inner boundary points. In Theorem 2.1, we will prove that for $d \geq 2$

$$\lim_{n\to\infty} \frac{M(n)}{\log n} = \frac{1}{-\log P(T_0 < T_b)} \quad \text{a.s.}$$

Here, T_a is the first time the random walk started at the origin hits a after time 0, and b is a neighbor site of the origin. To compare, a classical result of Erdős and Taylor [5] says that for $d \geq 3$,

$$\lim_{n\to\infty} \frac{M_0(n)}{\log n} = \frac{1}{-\log P(T_0 < \infty)} \quad \text{a.s.},$$

and for $d = 2$, $M_0(n)/(\log n)^2$ is bounded away from zero and infinity a.s. (the limit exists and is identified [2] as mentioned later in Section 2).

These results illuminate the geometric structure of the random walk range as well as the nature of recurrence or transience of random walks. We are able to infer that the favorite site is outside the inner boundary from some time onwards with probability one. This may appear intuitively clear; it seems improbable for the favorite point to continue to be an inner boundary point since it must be visited many times, but our result further shows that there are many inner boundary points that are visited many times, with amounts comparable to that of the favorite point for $d \geq 3$. In addition, the growth order of $M_0(n)$ is the same for all $d \geq 2$, meaning the phase transition which occurs between $d = 2$ and $d \geq 3$ for $M_0(n)$ does not occur for $M(n)$.

In Theorem 2.2, which is a strong claim in comparison to Theorem 2.1, we will provide an explicit answer to the question of how many frequently visited sites among the inner boundary points exist.

The upper bounds for both Theorems 2.1 and 2.2 are obtained using the idea in [5]. The Chebyshev inequality and the Borel–Cantelli lemma are also used in the same way as in [5]. On the other hand, $M(n)$ is not monotone, while $M_0(n)$ is monotone. We work with the walk and its trajectory at the times 2^k and find a process that is monotone and a bit larger than $M(n)$, but with the desired asymptotics.

On the other hand, the idea for the proof of the lower bound is different from that for the known results. In [2], a Brownian occupation measure was used in the proof. Rosen [11] provided another proof to the result of [2], in which he computed a crossing number instead of the number of the frequently visited site. In this paper, we use the Chebyshev inequality and the Borel–Cantelli lemma as in [11] but for the number of the frequently visited sites among the inner boundary points. In addition, as the proof of the upper bound, we estimate a number slightly smaller than the number of the frequently visited site among the inner boundary points.

We conclude this introduction by mentioning some known results about the inner boundary points of the random walk range that are closely related to the present subject. Let L_n be the