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A CLT for multi-dimensional martingale differences in a lexicographic order

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Dedicated to the memory of Mikhail Gordin

Abstract

We prove a central limit theorem for a square-integrable ergodic stationary multi-dimensional random field of martingale differences with respect to a lexicographic order.

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Keywords: Central limit theorem; Random fields; Multi-dimensional martingale differences; Lexicographic order

1. Introduction

M. Rosenblatt [10] stated a central limit theorem (CLT) for *ergodic* square-integrable stationary two-dimensional random fields of martingale differences, with lexicographic order, which was a step towards a CLT for random fields satisfying some strong mixing conditions. In order to formulate the assertion we start with the following notations.

Notations. On $\mathbb{N}^d,\ d \geq 2$, we take a *lexicographic order* as follows: $\boldsymbol{n} = (n^1, \dots, n^d) < \boldsymbol{m} = (m^1, \dots, m^d)$ if and only if $n^d < m^d$ or there exists $i = 1, \dots, d-1$, such that $n^j = m^j$ for $i < j \leq d$ and $n^i < m^i$. Let $\{\zeta_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ be a square integrable array of random variables and

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let $\mathcal{F}_{\boldsymbol{n}}$ be the σ-field generated by $\{\zeta_{\boldsymbol{m}} : \boldsymbol{m} \leq \boldsymbol{n}\}$. We say that $\{\zeta_{\boldsymbol{n}}\}$ is a *d-dimensional martingale difference with respect to* $\{\mathcal{F}_{\boldsymbol{n}}\}$ if $\mathbb{E}[\zeta_{\boldsymbol{n}}|\mathcal{F}_{\boldsymbol{m}}] = 0$ for every $\boldsymbol{m} < \boldsymbol{n}$.

Rosenblatt's assertion for the ergodic stationary martingale differences was that, for $d = 2$, $\frac{1}{\sqrt{mn}} \sum_{j=0}^{m} \sum_{\ell=0}^{n} \zeta_{j,\ell}$ converges in distribution to a normal law, as $\min\{m, n\} \to \infty$. An indication of proof was mentioned in [10]. We are interested in convergence of the above expression as $\max\{m, n\} \to \infty$. As we will explain, one needs ergodicity of the individual shifts; mere ergodicity of the random field is not sufficient in general for this convergence.

Huang [8] proved the CLT for ergodic stationary two-dimensional square-integrable martingale differences with the lexicographic order, in the particular case of $m = n$.

Dedecker [3] has a CLT for multi-dimensional stationary random fields with averaging along Følner sequences. Clearly $m \times n$ rectangles with $mn \to \infty$ need not have the Følner property, so Dedecker's result does not imply the above mode of convergence, even when corrected to assume ergodicity of the individual shifts. It is worth mentioning that Dedecker's result is quite general and yields a CLT without requiring ergodicity—in that case the limiting distribution is (as usual) a mixture of normal distributions.

The purpose of the following note is to give a simple proof of the CLT for a d-dimensional martingale difference as above $\{\zeta_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$. Specifically, for $d = 2$, to give a sufficient condition for convergence under the assumption $\max\{m, n\} \to \infty$. Under some moment conditions, a rate in the CLT is also given.

2. The CLT for multi-dimensional martingale differences

We prove below a CLT for the random field $\{\zeta_{\boldsymbol{n}}, \mathcal{F}_{\boldsymbol{n}}\}$ described above. The second part of our CLT below is new. The first part of the theorem is included for the sake of completeness.

From now on we use the following notation: $D_{\boldsymbol{n}} := \{\boldsymbol{m} : 0 \leq m^i < n^i, 1 \leq i \leq d\}$.

Since the adaptation of the notation and proofs from dimension two to any finite dimension $d > 2$ are straightforward, for the sake of clarity and in order to avoid too long expressions, we prove the relevant statements in dimension two.

Theorem 2.1. *Let $\{\zeta_{\boldsymbol{n}}, \mathcal{F}_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ be a square-integrable ergodic stationary d-dimensional random field of real martingale differences. Then $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m} \in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ converges in distribution to $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ as $\min\{n^1, n^2, \ldots, n^d\} \to \infty$.*

If the d shifts of the random field are ergodic, then $\frac{1}{\sqrt{n^1 \cdot n^2 \cdots n^d}} \sum_{\boldsymbol{m} \in D_{\boldsymbol{n}}} \zeta_{\boldsymbol{m}}$ converges in distribution to $\mathcal{N}(0, \mathbb{E}|\zeta_{\boldsymbol{0}}|^2)$ as $n^1 \cdot n^2 \cdots n^d \to \infty$ (equivalently, as $\max\{n^1, n^2, \ldots, n^d\} \to \infty$).

Proof. The first assertion is a consequence of Theorem 1 of Dedecker [3] (a result about stationary random fields with averaging along Følner sequences). We mention that this assertion can be proved also along the same lines of the proof of the second assertion given below. Only the multi-dimensional mean ergodic theorem is needed, instead of Lemma 2.2.

We prove the second assertion. For the sake of clarity we prove it for $d = 2$. First we make the following observation. A two-dimensional sequence of random variables $\{Z_{m,n}\}_{m,n \geq 1}$ converges in distribution, as $mn \to \infty$, to a random variable Z if and only if for every subsequences $\{m_k\}, \{n_k\}$ with $m_k n_k \to_k \infty$, $\{Z_{m_k,n_k}\}$ converges in distribution to Z. Indeed, this claim is about numerical sequences ($\{\mathbb{P}(Z_{m,n} \leq t)\}$ for fixed t a point of continuity of Z), and can be easily verified by definition.