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Stochastic Processes and their Applications 116 (2006) 539–567

stochastic processes and their applications

www.elsevier.com/locate/spa

A conditional limit theorem for tree-indexed random walk

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Received 14 March 2005; received in revised form 1 November 2005; accepted 1 November 2005
Available online 19 December 2005

Abstract

We consider Galton–Watson trees associated with a critical offspring distribution and conditioned to have exactly n vertices. These trees are embedded in the real line by assigning spatial positions to the vertices, in such a way that the increments of the spatial positions along edges of the tree are independent variables distributed according to a symmetric probability distribution on the real line. We then condition on the event that all spatial positions are nonnegative. Under suitable assumptions on the offspring distribution and the spatial displacements, we prove that these conditioned spatial trees converge as $n \to \infty$, modulo an appropriate rescaling, towards the conditioned Brownian tree that was studied in previous work. Applications are given to asymptotics for random quadrangulations.

Keywords: Galton–Watson tree; Tree-indexed random walk; Spatial tree; Conditioned tree; Conditioned Brownian snake; Invariance principle; ISE; Well-labelled tree; Random quadrangulations

1. Introduction

The main goal of the present work is to prove an invariance principle for tree-indexed random walks on the real line which are constrained to remain on the positive side. One major motivation for this problem came from recent asymptotic results for random quadrangulations which have been established by Chassaing and Schaeffer [8].

The asymptotic behavior of Galton–Watson trees conditioned to have a large fixed progeny was investigated by Aldous [1] in connection with the so-called Continuum Random Tree (CRT).

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doi:10.1016/j.spa.2005.11.008

Precisely, under the assumption that the offspring distribution μ is critical and has finite variance $\sigma^2 > 0$, a Galton–Watson tree conditioned to have exactly n vertices, with edges rescaled by the factor $\sigma n^{-1/2}/2$, will converge in distribution, in a suitable sense, towards the CRT. A convenient way of making this convergence mathematically precise is to use the contour function of the conditioned Galton–Watson tree (cf. Fig. 1 below). Modulo a rescaling analogous to the classical Donsker theorem for random walks, this contour function converges in distribution as $n \to \infty$ towards a normalized Brownian excursion, that is a positive Brownian excursion conditioned to have duration 1 (cf. the convergence of the first components in Theorem 2.1 below). Informally we may say that the normalized Brownian excursion is the contour function of the CRT. See [14] for analogous contour descriptions of the more general Lévy trees, and [13] for a recent generalization of Aldous' theorem.

In view of various applications, and in particular in connection with the theory of superprocesses, it is interesting to combine the branching structure of the Galton–Watson tree with spatial displacements. Here we consider the simple special case where these spatial displacements are given by a one-dimensional symmetric random walk on the real line with jump distribution γ. This means that i.i.d. random variables Y_e with distribution γ are associated with the different edges of the tree, and that the spatial position U_v of a vertex v is obtained by summing the displacements Y_e corresponding to edges e that belong to the path from the root to the vertex v. The resulting object, called here a spatial tree, consists of a (random) pair $(\mathcal{T}, U)$, where $\mathcal{T}$ is a discrete (plane) tree and U is a mapping from the set of vertices of $\mathcal{T}$ into $\mathbb{R}$. In the same way as the tree $\mathcal{T}$ can be coded by its contour function, a convenient way of encoding the spatial positions is via the spatial contour function (see Section 2 for a precise definition, and Fig. 2 for an example).

From now on, we suppose that the tree $\mathcal{T}$ is a Galton–Watson tree with offspring distribution μ satisfying the above assumptions, and conditioned to have exactly n vertices, and that the spatial positions U_v are generated as explained in the preceding paragraph. We assume furthermore that μ has (small) exponential moments and that $\gamma([x, \infty)) = o(x^{-4})$ as $x \to \infty$. We denote by ρ^2 the variance of γ. Then rescaling both the edges of $\mathcal{T}$ by the factor $\sigma n^{-1/2}/2$ and the spatial displacements U_v by $\rho^{-1}(\sigma/2)^{1/2} n^{-1/4}$ will lead as $n \to \infty$ to a limiting object which is independent of μ and γ. A precise statement for this convergence is given in Theorem 2.1 below, which is taken from Janson and Marckert [19] (see [8,16,26] for similar statements under different assumptions — related results have also been obtained by Kesten [20] under other conditionings of the tree). This convergence is closely related to the approximation of superprocesses by branching particle systems: see in particular [21]. Roughly speaking, the limiting object combines the branching structure of the CRT with spatial displacements given by independent linear Brownian motions along the edges of the tree. A convenient representation of this limiting object, which is used in Theorem 2.1, is provided by the Brownian snake (see e.g. [22]). To describe this approach, let $r \in \mathbb{R}$, which will represent the initial position (the spatial position of the root) and let $\mathbf{e} = (\mathbf{e}(s), 0 \le s \le 1)$ be a normalized Brownian excursion. Let $Z^r = (Z^r(s), 0 \le s \le 1)$ be a real-valued process such that, conditionally given $\mathbf{e}$, Z^r is Gaussian with mean and covariance given by

- $E[Z^r(s)] = r$ for every $s \in [0, 1]$;
- $\mathrm{cov}(Z^r(s), Z^r(s')) = \inf_{s \le t \le s'} \mathbf{e}(t)$ for every $0 \le s \le s' \le 1$.

Informally, each time $s \in [0, 1]$ corresponds via the contour function coding to a vertex of the CRT, and $Z^r(s)$ is the spatial position of this vertex. The formula for the conditional covariance of $Z^r(s)$ and $Z^r(s')$ is then justified by the fact that $\inf_{s \le t \le s'} \mathbf{e}(t)$ is the generation of the