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Time homogeneous diffusion with drift and killing to meet a given marginal

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Abstract

In this article, it is proved that for any probability law μ over $\mathbb{R}$ and a drift field $b : \mathbb{R} \to \mathbb{R}$ and killing field $k : \mathbb{R} \to \mathbb{R}_+$ which satisfy hypotheses stated in the article and a given terminal time $t > 0$, there exists a string m, an $\alpha \in (0, 1]$, an initial condition $x_0 \in \mathbb{R}$ and a process X with infinitesimal generator $\left(\frac{1}{2}\frac{\partial^2}{\partial m \partial x} + b\frac{\partial}{\partial m} - \frac{\partial K}{\partial m}\right)$ where $k = \frac{\partial K}{\partial x}$ such that for any Borel set $B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{P}(X_t \in B | X_0 = x_0) = \alpha\mu(B).$$

Firstly, it is shown the problem with drift and without killing can be accommodated, after a simple co-ordinate change, entirely by the proof in Noble (2013). The killing field presents additional problems and the proofs follow the lines of Noble (2013) with additional arguments.

Keywords: Time homogeneous gap diffusion; Drift; Killing; Kreĭn strings; Marginal distribution

1. Introduction

1.1. Results and method of proof

Let μ be a probability measure over $\mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R} \to \mathbb{R}_+$ given drift and killing functions. Set

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$$\widetilde{b}(x) = \begin{cases} b(x) & x \in \text{suppt}(\mu) \\ 0 & x \notin \text{suppt}(\mu), \end{cases} \qquad B(x) = \begin{cases} \int_{[0,x]} \widetilde{b}(y)dy & x \geq 0 \\ -\int_{[x,0)} \widetilde{b}(y)dy & x < 0 \end{cases} \tag{1}$$

where suppt(μ) denotes the support of the measure μ. Let

$$\widehat{k}(x) = \begin{cases} k(x) & x \in \text{suppt}(\mu) \\ 0 & x \notin \text{suppt}(\mu), \end{cases} \qquad K(x) = \begin{cases} \int_{[0,x]} \widehat{k}(y)dy & x \geq 0 \\ -\int_{[x,0)} \widehat{k}(y)dy & x < 0. \end{cases} \tag{2}$$

Hypothesis 1.1 (*Hypothesis on Drift b, Killing Field k and Measure* μ)**.** The target probability measure, drift and killing (μ, b, k) satisfy the following conditions.

1. B from (1) and K from (2) are absolutely continuous with respect to μ.
2. Let $l_-(x) = \sup\{y \in \text{suppt}(\mu) \cap (-\infty, x)\}$ and let $l_+(x) = \inf\{y \in \text{suppt}(\mu) \cap (x, +\infty)\}$, then

$$\sup_{x\in\mathbb{R}} \lim_{h\downarrow 0} \int_{l_-(x)-h}^{l_+(x)+h} |\widetilde{b}(x)|dx < 1 \tag{3}$$

   where $\widetilde{b}$ is from (1).
3. Let $c : (0, 1) \to \mathbb{R}_+$ denote the function defined by:

$$c(x) = \frac{\left(\ln \frac{1}{x}\right) - (1-x)}{(1-x)^2}. \tag{4}$$

   Let γ satisfy:

$$\gamma = \frac{1}{2}\left(1 - \sup_{x\in\mathbb{R}} \lim_{h\downarrow 0} \int_{l_-(x)-h}^{l_+(x)+h} |\widetilde{b}(x)|dx\right). \tag{5}$$

   Then (b, μ) satisfies:

$$\int_{-\infty}^{\infty} \left(\int_{0\wedge x}^{0\vee x} e^{F(b,y)}dy\right) \mu(dx) < +\infty \tag{6}$$

   where

$$F(b, y) = 2\left(\int_{0\wedge y}^{0\vee y} \left|\widetilde{b}(x)\right| dx + c(\gamma) \times \sup_{\underline{t}:(0\wedge y)=t_0<\cdots<t_n=(0\vee y)} \sum_{i=0}^{n-1} \left\{\left(\int_{t_i}^{t_{i+1}} |\widetilde{b}(x)|dx\right)^2\right\}\right) \tag{7}$$

   and $\widetilde{b}$ is defined by (1). Here the maximum is taken over sequences of length n for all $n \in \mathbb{N}$.
4. $\lim_{x\to\pm+\infty} \frac{\partial K}{\partial \mu}(x) = 0$.

   Let $z_+ = \sup\{x \in \text{suppt}(\mu)\}$ and $z_- = \inf\{x \in \text{suppt}(\mu)\}$. Then $\frac{\partial K}{\partial \mu}(x)$ is defined to be 0 for $x > z_+$ and $x < z_-$.