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Moment bounds and asymptotics for the stochastic wave equation

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Abstract

We consider the stochastic wave equation on the real line driven by space–time white noise and with irregular initial data. We give bounds on higher moments and, for the hyperbolic Anderson model, explicit formulas for second moments. These bounds imply weak intermittency and allow us to obtain sharp bounds on growth indices for certain classes of initial conditions with unbounded support.

MSC: primary 60H15; secondary 60G60; 35R60

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1. Introduction

In this paper, we will study the following stochastic wave equation:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \kappa^2 \dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x), & x \in \mathbb{R},\ t \in \mathbb{R}_+^*, \\ u(0,\circ) = g(\circ), \qquad \dfrac{\partial u}{\partial t}(0,\circ) = \mu(\circ), \end{cases} \tag{1.1}$$

where $\mathbb{R}_+^* =]0,\infty[$, $\dot{W}$ is space–time white noise, $\rho(u)$ is globally Lipschitz, $\kappa > 0$ is the speed of wave propagation, g and μ are the (deterministic) initial position and velocity, respectively,

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and $\circ$ denotes the spatial dummy variable. The linear case, $\rho(u) = \lambda u, \lambda \neq 0$, is called *the hyperbolic Anderson model* [19]. If $\rho(u) = \lambda\sqrt{\varsigma^2 + u^2}$, then we call (1.1) the *near-linear Anderson model.*

This equation has been extensively studied during last two decades by many authors: see e.g., [4,6,7,32,37] for some early work, [16,18,37] for an introduction, [19,20] for asymptotic properties of moments, [12,15,17,21,26–28,33–35] for the stochastic wave equation in the spatial domain $\mathbb{R}^d$, $d > 1$, [22,36] for regularity of the solution, [2,3] for the stochastic wave equation with values in Riemannian manifolds, and [11,30,31] for wave equations with polynomial nonlinearities.

In this paper, we consider initial data with very little regularity. In particular, we assume that the initial position g belongs to $L^2_{loc}(\mathbb{R})$, the set of locally square integrable Borel functions, and the initial velocity μ belongs to $\mathcal{M}(\mathbb{R})$, the set of locally finite Borel measures. Denote the solution to the homogeneous equation by

$$J_0(t,x) := \frac{1}{2}\left(g(x+\kappa t) + g(x-\kappa t)\right) + (\mu * G_\kappa(t,\circ))(x), \tag{1.2}$$

where

$$G_\kappa(t,x) = \frac{1}{2} H(t) 1_{[-\kappa t, \kappa t]}(x)$$

is the wave kernel function. Here, $H(t)$ is the Heaviside function (i.e., $H(t) = 1$ if $t \geq 0$ and 0 otherwise), and "$*$" denotes convolution in the space variable. Regarding the stochastic pde (spde) (1.1), we interpret it in the integral (mild) form:

$$u(t,x) = J_0(t,x) + I(t,x),$$

where

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}} G_\kappa\left(t-s, x-y\right) \rho\left(u\left(s,y\right)\right) W\left(\mathrm{d}s, \mathrm{d}y\right).$$

We call $I(t,x)$ the *stochastic integral part* of the random field solution. This stochastic integral is interpreted in the sense of Walsh [37].

The first contribution of this paper concerns estimates and exact formulas for moments of the random field solution to (1.1) (for the stochastic *heat* equation, this type of result has recently been obtained in [9]). Consider, for instance, the case where $\rho(u)^2 = \lambda^2(\varsigma^2 + u^2)$ for some λ and $\varsigma \in \mathbb{R}$, and let $I_n(\cdot)$ be the modified Bessel function of the first kind of order n, or simply the *hyperbolic Bessel function* [29, 10.25.2, p. 249]:

$$I_n(x) := \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(x^2/4\right)^k}{k!\Gamma(n+k+1)}, \tag{1.3}$$

(see [25,38] for its relation with the wave equation). Define two kernel functions $\mathcal{K}(t,x) := \mathcal{K}(t,x;\kappa,\lambda)$ and $\mathcal{H}(t) := \mathcal{H}(t;\kappa,\lambda)$ as follows:

$$\mathcal{K}(t,x;\kappa,\lambda) := \begin{cases} \dfrac{\lambda^2}{4} I_0\left(\sqrt{\dfrac{\lambda^2\left((\kappa t)^2 - x^2\right)}{2\kappa}}\right) & \text{if } -\kappa t \leq x \leq \kappa t, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$