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Self-similarity and spectral asymptotics for the continuum random tree

David Croydon[a,*], Ben Hambly[b]

[a] *Department of Statistics, University of Warwick, Coventry, CV4 7AL, UK*
[b] *Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford, OX1 3LB, UK*

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Abstract

We use the random self-similarity of the continuum random tree to show that it is homeomorphic to a post-critically finite self-similar fractal equipped with a random self-similar metric. As an application, we determine the mean and almost-sure leading order behaviour of the high frequency asymptotics of the eigenvalue counting function associated with the natural Dirichlet form on the continuum random tree. We also obtain short time asymptotics for the trace of the heat semigroup and the annealed on-diagonal heat kernel associated with this Dirichlet form.

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Keywords: Continuum random tree; Self-similar fractal; Spectral asymptotics; Heat kernel

1. Introduction

One of the reasons the continuum random tree of Aldous has attracted such great interest is that it connects together a number of diverse areas of probability theory. On one hand, it appears from discrete probability as the scaling limit of combinatorial graph trees and probabilistic branching processes; and on the other hand, it is intimately related to a continuous time process, namely the normalised Brownian excursion, [1]. However, with both of these representations of the continuum random tree, there does not appear to be an obvious description of the structure

* Corresponding author. Tel.: +44 0 24 7657 4812; fax: +44 0 24 7652 4532.
E-mail addresses: d.a.croydon@warwick.ac.uk (D. Croydon), hambly@maths.ox.ac.uk (B. Hambly).

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of the set itself. In this paper, we demonstrate that the continuum random tree has a recursive description as a random self-similar fractal, and show that the set is always homeomorphic to a deterministic subset of the Euclidean plane. As an application of this precise description of the random self-similarity of the continuum random tree, we deduce results about the spectrum and on-diagonal heat kernel of the natural Dirichlet form on the set, using techniques developed for random recursive self-similar fractals.

From its graph tree scaling limit description, Aldous showed how the continuum random tree has a certain random self-similarity, [2]. In this article, we use this result iteratively to label the continuum random tree, $\mathcal{T}$, using a shift space over a three letter alphabet. This enables us to show that there is an isometry from $\mathcal{T}$, with its natural metric $d_{\mathcal{T}}$ (see Section 2 for a precise definition of $\mathcal{T}$ and $d_{\mathcal{T}}$, and Section 3 for the decomposition of $\mathcal{T}$ that we apply), to a deterministic subset of $\mathbb{R}^2$, T say, equipped with a random metric R, **P**-a.s., where **P** is the probability measure on the probability space upon which all the random variables of the discussion are defined. This metric is constructed using random scaling factors in an adaptation of the now well-established techniques of [3] for building a resistance metric on a post-critically finite self-similar fractal. We note that on a tree, the resistance and geodesic metrics are the same. Furthermore, we show that the isometry in question also links the natural Borel probability measures on the spaces $(\mathcal{T}, d_{\mathcal{T}})$ and (T, R). The relevant measures will be denoted by μ and μ^T respectively, with μ arising as the scaling limit of the uniform measures on the graph approximations of $\mathcal{T}$ (see [1], for example), and μ^T being the random self-similar measure that is associated with the construction of R. The result that we prove is the following; full descriptions of (T, R, μ^T) are given in Section 4, and the isometry is defined in Section 5.

Theorem 1. *There exists a deterministic post-critically finite self-similar dendrite, T, equipped with a (random) self-similar metric, R, and Borel probability measure, μ^T, such that (T, R, μ^T) is equivalent to $(\mathcal{T}, d_{\mathcal{T}}, \mu)$ as a measure-metric space,* **P***-a.s.*

Previous analytic work on the continuum random tree in [4] obtained estimates on the quenched and the annealed heat kernel for the tree. We can now adapt some techniques of [5] to consider the spectral asymptotics of the tree. As a by-product, we are also able to refine the results on the annealed heat kernel to show the existence of a short time limit for $t^{2/3}\mathbf{E}p_t(\rho, \rho)$ at the root of the tree ρ, where the notation **E** is used to represent expectation under the probability measure **P**.

The natural Dirichlet form on $L^2(\mathcal{T}, \mu)$ may be thought of simply as the electrical energy when we consider $(\mathcal{T}, d_{\mathcal{T}})$ as a resistance network. We shall denote this form by $\mathcal{E}_{\mathcal{T}}$, and its domain as $\mathcal{F}_{\mathcal{T}}$, and explain in Section 2 how it may be constructed using results of [6]. The eigenvalues of the triple $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$ are defined to be the numbers λ which satisfy

$$\mathcal{E}_{\mathcal{T}}(u, v) = \lambda \int_{\mathcal{T}} uv\,\mathrm{d}\mu, \quad \forall v \in \mathcal{F}_{\mathcal{T}} \tag{1}$$

for some eigenfunction $u \in \mathcal{F}_{\mathcal{T}}$. The corresponding eigenvalue counting function, N, is obtained by setting

$$N(\lambda) := \#\{\text{eigenvalues of } (\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu) \leq \lambda\}, \tag{2}$$

and we prove in Section 6 that this is well-defined and finite for any $\lambda \in \mathbb{R}$, **P**-a.s. In Section 6, we also prove the following result, which shows that asymptotically, the mean and **P**-a.s. behaviour of N are identical.