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Moments and distribution of the local time of a two-dimensional random walk

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Abstract

Let $\ell(n, x)$ be the local time of a random walk on $\mathbb{Z}^2$. We prove a strong law of large numbers for the quantity $L_n(\alpha) = \sum_{x \in \mathbb{Z}^2} \ell(n, x)^\alpha$ for all $\alpha \geq 0$. We use this result to describe the distribution of the local time of a typical point in the range of the random walk.

Key words: Local time; Random walk; Strong law of large numbers

1. Introduction

Let $X_i, i \in \mathbb{N}$, be a sequence of i.i.d. random vectors on some probability space $(\Omega, \mathbb{P})$, which have values in $\mathbb{Z}^2$, mean 0, and a finite non-singular covariance matrix Σ. We write

$$S_0 := 0, \qquad S_n := \sum_{i=1}^{n} X_i, \quad n \geq 1, \tag{1}$$

for a $\mathbb{Z}^2$-valued random walk. Let $\ell(n, x)$ be its local time,

$$\ell(n, x) := \sum_{i=0}^{n} \mathbb{1}\{S_i = x\}, \quad x \in \mathbb{Z}^2. \tag{2}$$

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We will always assume that the characteristic function of X_i,

$$\chi(k) := \mathbb{E}\exp(\mathrm{i}\langle k, X_1\rangle), \quad k \in J := [-\pi, \pi)^2, \tag{3}$$

satisfies $\chi(k) = 1 \Leftrightarrow k = 0$. Here $\langle\cdot,\cdot\rangle$ stands for the standard scalar product in $\mathbb{R}^2$.

In this paper we prove the following strong law of large numbers for random variables

$$L_n(\alpha) := \sum_{x\in\mathbb{Z}^2} \ell(n, x)^\alpha, \quad \alpha \geq 0, n \in \mathbb{N}. \tag{4}$$

Theorem 1. *For all $\alpha \geq 0$, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \frac{L_n(\alpha)}{n(\log n)^{\alpha-1}} = \frac{\Gamma(\alpha+1)}{(2\pi\sqrt{\det\Sigma})^{\alpha-1}}. \tag{5}$$

Remark. This result is trivial for $\alpha = 1$ and well known for $\alpha = 0$. In the second case, $L_n(0) = \sum_x \mathbb{1}\{\ell(n, x) \geq 1\} =: R(n)$ is the size of the range of the random walk. For the simple random walk it was proved in [5] that the range satisfies

$$\lim_{n\to\infty} \frac{\log n}{n} R(n) = \pi, \quad \mathbb{P}\text{-a.s.} \tag{6}$$

For a non-simple walk with a covariance matrix Σ the right-hand side of (6) must be multiplied by $2\sqrt{\det\Sigma}$.

There are at least two reasons why the quantity $L_n(\alpha)$ is worth studying. First, if α is an integer, then $L_n(\alpha)$ is related to the number of α-fold self-intersections of the random walk (see also (11) below). This is of great importance, mainly with $\alpha = 2$ or $\alpha = 0$, for the so-called self-interacting random walk; see e.g. [4]. In this paper, however, we do not require α to be integer. $L_n(\alpha)$ can be then considered as a possible candidate for a definition of the number of α-fold self-intersections for all real positive α.

The second related subject, which was the original motivation for studying $L_n(\alpha)$, is the so-called *random walk in random scenery* and with it closely connected problem of *aging in trap models*. We describe this problem briefly. Let τ_x, $x \in \mathbb{Z}^2$, be a collection of i.i.d. random variables independent of X_i. Define

$$Z_n := \sum_{i=0}^{n} \tau_{S_i}. \tag{7}$$

This process (called usually random walk in random scenery) was considered for the first time for one-dimensional random walks in [8]. Two-dimensional walks were studied in [2], where the random scenery τ_x was required to have mean zero and a finite variance σ^2. It was proved there that the process $Z_{\lfloor nt\rfloor}/\sqrt{n\log n}$ converges to the standard Brownian motion with a variance depending on σ and Σ.

In [1] we needed to control the behaviour of Z_n for a scenery τ_x in the domain of attraction of a non-negative, α-stable, $\alpha \in (0, 1)$, law. The interest in this kind of scenery originated in the study of aging in the so-called Bouchaud trap model. This model was proposed in the physics literature [3] to explain basic mechanisms that can be responsible for peculiar dynamical properties (like aging) of complex disordered systems. The α-stable sceneries with small α correspond to the low-temperature regime in these systems that is particularly interesting. In