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Infinite horizon stopping problems with (nearly) total reward criteria

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Abstract

We study an infinite horizon optimal stopping Markov problem which is either undiscounted (total reward) or with a general Markovian discount rate. Using ergodic properties of the underlying Markov process, we establish the feasibility of the stopping problem and prove the existence of optimal and ε-optimal stopping times. We show the continuity of the value function and its variational characterisation (in the viscosity sense) under different sets of assumptions satisfied by large classes of diffusion and jump–diffusion processes. In the case of a general discounted problem we relax a classical assumption that the discount rate is uniformly separated from zero.

Keywords: Optimal stopping; Infinite horizon; Total reward; General Markovian discounting; Non-uniformly ergodic Markov processes

1. Introduction

The theory of optimal stopping has recently seen its renaissance due to applications in finance and operations research (e.g., pricing of American options, optimal timing of a sale or valuation of natural resources, see [23] and references therein, or pricing of swing options with applications in energy trading [3]), and statistics (e.g., sequential hypothesis testing, see [22,26]

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and references therein). In some applications, such as the valuation of American options, there is a natural bound for stopping times—the maturity date of the option. This results in finite horizon stopping problems. In others, like finding an optimal time to sell a stock/business or valuing natural resources, the horizon is infinite leading to an optimal stopping problem of the following form:

$$w(x) = \sup_{\tau} \limsup_{T\to\infty} \mathbb{E}^x \Big\{ \int_0^{\tau\wedge T} e^{-\int_0^s r(X_u)du} f(X_s)ds + e^{-\int_0^{\tau\wedge T} r(X_u)du} g(X_{\tau\wedge T}) \Big\}. \tag{1}$$

Here, the stochastic process $X(t)$ models the underlying randomness in the world, such as the price of a stock, the price of a natural resource (gas, oil, etc.), or factors influencing the interest rate. The running profit/cost is represented by f while the proceeds from the final sale at time τ or the closure of the production facility are given by g. The function r corresponds to an instantaneous interest rate that is used for discounting of future cash-flows. For years, it had seemed sensible to assume that interest rates were uniformly separated from 0. However, recent developments in Japan and in Europe showed that (even nominal) interest rates for government bonds can get arbitrarily close to 0.

The uniform separation of the discount rate from 0 (i.e., $\inf_x r(x) > 0$) is in line with a large strand of literature in infinite-horizon optimal stopping [2,22–24]; commonly, the discount rate is a positive constant. This ensures, under appropriate assumptions on the growth of f and g, that the value function is finite and allows to approximate the infinite horizon stopping problem with finite horizon ones. An abolition of the discounting or a relaxation of its separation from zero brings in a lot of difficulties. In particular, the integrability in the functional is at risk; in Section 2 we show that this is indeed the case. A solution to this problem is to impose restrictive assumptions on f and g. In martingale approaches it is common to assume that $\mathbb{E}^x\{\int_0^\infty e^{-\int_0^s r(X_u)du} f^-(X_s)ds\} < \infty$ for every x and that the family $\{e^{-\int_0^\tau r(X_u)du} g^-(X_\tau)$: τ-a stopping time}, is uniformly integrable, see [20,22] and references therein. Alternatively, one can take f to be non-positive (then the integral term penalises for waiting only) [22,26].

A general stopping problem without the restrictions on f was studied in [25,27,29] for uniformly geometrically ergodic Feller–Markov processes. Such processes converge exponentially fast to their invariant measure and the speed of this convergence is independent of the starting point. Examples are usually constrained to processes on compact state spaces. An attempt to generalise these results was made in [28], where the author assumed that trajectories of the process can be split into excursions with square integrable lengths between two compact sets. In this setting optimality was studied within a narrow class of stopping rules.

The aim of this paper is to analyse the optimal stopping problem (1) for *non-uniformly ergodic* Feller–Markov processes with minimal assumptions on f and g, and a general discount rate r which is only assumed to be non-negative. Specifically, assume that the state of the world is described by a right-continuous time homogeneous Markov process $\big(X(t)\big)$ defined on a locally compact separable space E endowed with a metric ρ with respect to which every closed ball is compact. The Borel σ-algebra on E is denoted by $\mathcal{E}$. Let P_t be the semigroup generated by the process (X_t), i.e., $P_t\phi(x) = \mathbb{E}^x\{\phi(X_t)\}$ for any bounded measurable function $\phi : E \to \mathbb{R}$. The transition measure is given by $P_t(x, A) = \mathbb{P}^x\{X_t \in A\}$ for $A \in \mathcal{E}$.

We make the following standing assumptions:

(A1) (Weak Feller property)

$$P_t\, \mathcal{C}_0 \subseteq \mathcal{C}_0,$$

where $\mathcal{C}_0$ is the space of continuous bounded functions $E \to \mathbb{R}$ vanishing in infinity.