Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 124 (2014) 3921–3947

stochastic processes and their applications

www.elsevier.com/locate/spa

Probabilistic interpretation for a system of quasilinear parabolic partial differential equation combined with algebra equations[☆]

Zhen Wu, Zhiyong Yu[*]

School of Mathematics, Shandong University, Jinan 250100, China

Received 8 July 2013; received in revised form 22 May 2014; accepted 11 July 2014
Available online 18 July 2014

Abstract

In this paper, we study a kind of system of second order quasilinear parabolic partial differential equation combined with algebra equations. Introducing a family of coupled forward–backward stochastic differential equations, and by virtue of some delicate analysis techniques, we give a probabilistic interpretation for it in the viscosity sense.

MSC: 60H10; 35K59; 35C99

Keywords: Forward–backward stochastic differential equation; Monotonicity condition; Parabolic partial differential equation; Viscosity solution

1. Introduction

In this paper, we are interested in the problem to provide a probabilistic interpretation of the solutions of the following system of quasilinear parabolic partial differential equation (PDE

☆ This work is supported by the National Natural Science Foundation of China (11101242, 11221061 and 61174092), the National Science Fund for Distinguished Young Scholars of China (11125102), 111 project (B12023), the Science and Technology Project of Shandong Province (2013GRC32201) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.
* Corresponding author. Tel.: +86 53188363470.
E-mail addresses: wuzhen@sdu.edu.cn (Z. Wu), yuzhiyong@sdu.edu.cn (Z. Yu).

http://dx.doi.org/10.1016/j.spa.2014.07.013

hereafter) combined with algebra equations:

$$\begin{cases} \partial_t u(t,x) + (\mathcal{L}u)(t,x,u(t,x),v(t,x)) + g(t,x,u(t,x),v(t,x)) = 0, \\ v(t,x) = \nabla u(t,x)\sigma(t,x,u(t,x),v(t,x)), \\ u(T,x) = \Phi(x), \quad (t,x) \in [0,T] \times \mathbb{R}^n, \end{cases} \tag{1}$$

where $\mathcal{L}$ is an infinitesimal operator defined by

$$(\mathcal{L}\phi)(t,x,y,z) := \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^\top)_{ij}(t,x,y,z)\frac{\partial^2\phi}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{n} b_i(t,x,y,z)\frac{\partial\phi}{\partial x_i}(t,x).$$

From the probabilistic viewpoint, the above PDE system should be related to a family of coupled forward–backward stochastic differential equations (FBSDEs) parameterized by $(t,x) \in [0,T] \times \mathbb{R}^n$ as follows:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds + \sigma(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})dW_s, \\ -dY_s^{t,x} = g(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x})ds - Z_s^{t,x}dW_s, \\ X_t^{t,x} = x, \qquad Y_T^{t,x} = \Phi(X_T^{t,x}), \quad s \in [t,T]. \end{cases} \tag{2}$$

How to coincide the solution of the PDE system (1) with that of the FBSDE systems (2) is an open problem posed by Peng [22] in 1999.

Coupled FBSDE in the form of (2) was first studied by Antonelli [1]. In his work, a local existence and uniqueness result was obtained. For the global existence and uniqueness results, there exist two main methods. One concerns a kind of *four-step scheme* approach introduced by Ma, Protter and Yong [13] which can be regarded as a sort of combination of the methods of PDE and probability theory. In this method, the diffusion coefficient σ of the forward equation is required to be non-degenerate and the coefficients are not allowed to be random. The second method is probabilistic. Under some monotonicity assumptions, Hu and Peng [10] obtained an existence and uniqueness result when the forward and backward equations have same dimensions. Peng and Wu [23] extended the result of [10] to the FBSDEs with forward and backward components of different dimensions and weakened the monotonicity assumptions. Yong [28,29] called the method used in [10,23] *method of continuation*. He introduced the notions of *bridge* and *Lyapunov operator* to make the method more systematic. Recently, Ma, Wu, Zhang and Zhang [14] proposed a *unified approach*, which can be regarded as a combination of existing methods.

It is classical that a system of first order semilinear PDEs can be solved via the method of characteristic curves (see Courant and Hilbert [6]). The well known Feynman–Kac formula gives a probabilistic interpretation for linear second order PDEs of elliptic or parabolic type, and has been generalized to the case of semilinear second order PDEs by Peng [19–21], Pardoux and Peng [16], Barles, Buckdahn and Pardoux [3], Pardoux, Pradeilles and Rao [17], Pardoux [15], Kobylanski [11], Buckdahn and Li [5], Wu and Yu [27], Buckdahn, Huang and Li [4] and so on, with the help of the theory of backward stochastic differential equations (BSDEs). In 1999, Pardoux and Tang [18] connected a special kind of coupled FBSDEs with quasilinear parabolic PDEs, and gave an existence result of the viscosity solution (see Crandall, Ishii and Lions [7]) under some monotonicity conditions which are different from [10,23]. However, in their paper the diffusion coefficient σ of the forward equation is independent of the representation term involved in the martingale part of the backward equation. (The representation term is denoted by the generic letter Z, as usual in the BSDEs theory.)