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A class of asymptotically self-similar stable processes with stationary increments

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Abstract

We generalize the BM-local time fractional symmetric α-stable motion introduced in Cohen and Samorodnitsky (2006) by replacing the local time with a general continuous additive functional (CAF). We show that the resulting process is again symmetric α-stable with stationary increments. Depending on the CAF, the process is either self-similar or lies in the domain of attraction of the BM-local time fractional symmetric α-stable motion. We also show that the process arises as a weak limit of a discrete "random rewards scheme" similar to the one described by Cohen and Samorodnitsky.

MSC: 60G18; 60G52; 60G50

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1. Introduction

α-stable self-similar processes with stationary increments (or α-stable sssi processes, for short) are attractive theoretical models for various natural phenomena exhibiting both heavy-tailed marginal distributions and invariant statistical behavior under suitable scaling. Recall that a stochastic process $\{X(t), t \geq 0\}$ is called α-stable if its finite-dimensional distributions are

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multivariate α-stable, and self-similar with index $H > 0$ (or simply H-self-similar) if

$$\{X(ct), t \geq 0\} \stackrel{d}{=} \{c^H X(t), t \geq 0\}$$

for any $c > 0$. Stationarity of increments means that

$$\{X(t + c) - X(t), t \geq 0\} \stackrel{d}{=} \{X(t) - X(0), t \geq 0\}$$

for any $c > 0$. There is an extensive literature on α-stable sssi processes; we refer the reader to [15, Chapter 7] and [9, Chapter 3] for introductory expositions and references.

In the Gaussian case, which corresponds to $\alpha = 2$, fractional Brownian motions and their constant multiples are known to be the only non-trivial sssi processes; see, for example, [15, Section 7.2]. More precisely, for any given index of self-similarity $H \in (0, 1)$, there is a unique Gaussian H-sssi process up to multiplicative constants, namely the fractional Brownian motion with index H. (There are no non-degenerate Gaussian H-sssi processes with $H \geq 1$.)

In contrast, when $0 < \alpha < 2$, there are typically many different α-stable H-sssi processes for any given feasible pair of indices (α, H). The feasible range of the pair (α, H) is given by

$$\begin{cases} 0 < H \leq 1/\alpha & \text{if } 0 < \alpha \leq 1, \\ 0 < H < 1 & \text{if } 1 < \alpha < 2. \end{cases}$$

Classification and understanding of α-stable sssi processes for $0 < \alpha < 2$ is an ongoing and fruitful project. Well-known examples of such processes include α-stable Lévy motions, linear fractional α-stable motions introduced in [16,11], and real harmonizable fractional α-stable motions introduced in [5]. The linear and real harmonizable α-stable motions are defined for $0 < \alpha \leq 2$, $0 < H < 1$, and both reduce to the fractional Brownian motion in the case $\alpha = 2$. The α-stable Lévy motion is defined for $0 < \alpha \leq 2$, $H = 1/\alpha$, and reduces to the Brownian motion in the case $\alpha = 2$.

In [7], Cohen and Samorodnitsky constructed a new class of symmetric α-stable (SαS) sssi processes. The focus on *symmetric* α-stable distributions was for simplicity only, and we will adopt the same convention in this paper. Their construction is based on the local time process of a fractional Brownian motion with index of self-similarity H, so the authors called their model the *FBM-H-local time fractional* SαS *motion*. They also showed that, in the case $H = 1/2$, this model arises naturally as a limiting process in a situation where many users perform independent symmetric random walks on distinct copies of the integer line and collect i.i.d. heavy-tailed random rewards associated with the integers that they visit. As the number of users increases, the properly normalized and time-scaled total reward process of all users converges weakly to the FBM-1/2-local time fractional SαS motion, which can also be called the BM-local time fractional SαS motion.

This paper extends the construction of Cohen and Samorodnitsky [7] in the case $H = 1/2$, by considering a general *continuous additive functional* of the Brownian motion instead of the Brownian local time. Following the authors' terminology, this model can be called the *BM-CAF fractional* SαS *motion*, where CAF stands for continuous additive functional. CAFs of Brownian motion can be thought of as generalizations of the local time concept, as they include the local time as a special case. This suggests that the BM-CAF fractional SαS motion will be similar in structure to the BM-local time fractional SαS motion, and in particular, it will be a natural approximating model for a generalized version of the random rewards scheme described in [7]. Our aim is to show that this is indeed the case. We will formally introduce the BM-CAF fractional SαS motion, explore its similarities and differences with the BM-local time fractional