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Stochastic Processes and their Applications 124 (2014) 4050–4079

stochastic processes and their applications

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Optimal expulsion and optimal confinement of a Brownian particle with a switching cost

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Received 6 February 2013; received in revised form 8 April 2014; accepted 16 July 2014
Available online 23 July 2014

Abstract

We solve two stochastic control problems in which a player tries to minimize or maximize the exit time from an interval of a Brownian particle, by controlling its drift. The player can change from one drift to another but is subject to a switching cost. In each problem, the value function is written as the solution of a free boundary problem involving second order ordinary differential equations, in which the unknown boundaries are found by applying the principle of smooth fit. For both problems, we compute the value function, we exhibit the optimal strategy and we prove its generic uniqueness.

MSC: primary 93E20; secondary 60G40; 60J65

Keywords: Stochastic control with switching cost; Principle of smooth fit; Free boundary problems; Martingale method

1. Description of the problem

Consider a game in which the player's goal is to force a Brownian particle out of an interval (say [0, 1]) as quickly as possible. At each instant, the player selects one of two opposite constant forces, either upwards or downwards, which adds or subtracts a constant drift μ to the Brownian motion. The player is allowed to switch between the two forces at any time, but at each switch,

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[1] The first author was partially supported by the Swiss National Foundation for Scientific Research.

http://dx.doi.org/10.1016/j.spa.2014.07.016

he incurs a penalty of c units of time ($c > 0$). The goal is to find a strategy that minimizes the expected penalized time, that is, the sum of the time needed for the particle to exit the interval and the switching penalties ("optimal expulsion problem").

We also solve the "opposite" problem, in which the goal is to keep the particle inside the interval for as long as possible, subject to the same kind of switching penalty, which is now subtracted from the time to exit the interval ("optimal confinement problem").

These two abstract problems can be viewed in the context of various applications, such as maintaining an inventory between certain bounds by controlling the production rate [9,22], or maintaining an insurance company's capital reserve between two bounds by controlling the insurance premium [2]. In certain asymptotic limits, these quantities may behave like a Brownian motion, and a change of production rate or of premium may entail a switching cost. The two boundaries may represent certain levels that one may want to reach as soon (or as late) as possible.

The presence of the switching cost is the key issue here: for instance, Prokhorov [21] solves a similar problem but without cost penalty, and Mandl [15] treats a control problem for a Brownian motion under a constraint on the number of switchings. When there is no switching cost, then the solutions of these problems are well-known (see [10, p.167–168]).

There is a well-developed literature for studying this kind of stochastic control problem, including [8,10,12,16,26]. Most frequently, these problems involve terminal costs and running costs. More recently, even more general kinds of performance criteria have been considered, as in [17], where the criterion also involves the running maximum of the observed process. In the presence of a switching cost, the problem falls into the theory of impulse control, as described for instance in [16, Chapter 6].

In order to solve our two control problems, we begin by formulating a free boundary problem for the value function. This involves splitting the state space into two regions, a *continuation* region and a *switching* region. The particular form of the regions is guessed from the description of the problem. In the continuation region, the value function solves certain ordinary differential equations, and in the complement of this region, the value function satisfies a relationship related to the switching cost. There are also boundary conditions at the extremities of the interval. In general, this system of equations is *not* sufficient to characterize the value function, and this is indeed the case here: it is necessary to specify appropriate additional conditions at the free boundaries between the regions, which we do using the so-called *principle of smooth fit* (see for example [19, p.147] and [20, Section 5.3.4]).

This approach has a long history, going back to [4,11,23,24], and, more recently, [19], and has proved to be quite successful in a wide range of problems, including, in addition to those in the references just mentioned, the problem of optimal switching (without cost) between two Brownian motions [14]. Other examples of optimal switching problems related to ours and with explicit solutions can be found in [1,6,13]. The problems considered in these papers differ from ours in particular because the state space is either the real line or the half-line, and there are no boundary conditions.

In our problem, if the switching cost is high enough, then, obviously, one should switch drifts rarely or not at all, and in fact, it turns out that there is a critical value $c^*(\mu)$, which turns out to be the same in both the expulsion and confinement problems, above which it is optimal never to switch drifts. We compute this value explicitly, and then we show that for costs $c < c^*(\mu)$, the optimal strategy is determined by four thresholds that are the endpoints of the switching regions. We determine these thresholds explicitly, up to the resolution of a single transcendental equation (in each problem).