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On the independence of the value function for stochastic differential games of the probability space

N.V. Krylov*

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455, United States

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Abstract

We show that the value function in a stochastic differential game does not change if we keep the same space $(\Omega, \mathcal{F})$ but introduce probability measures by means of Girsanov's transformation *depending* on the policies of the players. We also show that the value function does not change if we allow the driving Wiener processes to depend on the policies of the players. Finally, we show that the value function does not change if we perform a random time change with the rate depending on the policies of the players.

MSC: 49N70; 35D40; 49L25

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1. Introduction

Let $\mathbb{R}^d = \{x = (x^1, \dots, x^d)\}$ be a d-dimensional Euclidean space and let $d_1 \geq d$ be an integer. Assume that we are given separable metric spaces A and B, and let, for each $\alpha \in A$, $\beta \in B$, the following functions on $\mathbb{R}^d$ be given:

(i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(x) = \sigma(\alpha, \beta, x) = (\sigma^{\alpha\beta}_{ij}(x))$,
(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(x) = b(\alpha, \beta, x) = (b^{\alpha\beta}_i(x))$, and
(iii) real-valued functions $c^{\alpha\beta}(x) = c(\alpha, \beta, x) \geq 0$, $f^{\alpha\beta}(x) = f(\alpha, \beta, x)$, and $g(x)$.

* Tel.: +1 612 625 8338.
E-mail address: krylov@math.umn.edu.

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Under natural assumptions which will be specified later, on a probability space $(\Omega, \mathcal{F}, P)$ carrying a d_1-dimensional Wiener process w_t one associates with these objects and a bounded domain $G \subset \mathbb{R}^d$ a stochastic differential game with the diffusion term $\sigma^{\alpha\beta}(x)$, drift term $b^{\alpha\beta}(x)$, discount rate $c^{\alpha\beta}(x)$, running cost $f^{\alpha\beta}(x)$, and the final cost $g(x)$ payed when the underlying process first exits from G.

After the order of players is specified in a certain way it turns out (see our Remark 2.2) that the value function $v(x)$ of this differential game is a unique continuous in $\bar{G}$ viscosity solution of the Isaacs equation

$$H[v] = 0 \tag{1.1}$$

in G with boundary condition $v = g$ on ∂G, where for a sufficiently smooth function $u = u(x)$

$$H[u](x) = \sup_{\alpha \in A} \inf_{\beta \in B} [L^{\alpha\beta} u(x) + f^{\alpha\beta}(x)], \tag{1.2}$$

$$L^{\alpha\beta} u(x) := a_{ij}^{\alpha\beta}(x) D_{ij} u(x) + b_i^{\alpha\beta}(x) D_i u(x) - c^{\alpha\beta}(x) u(x),$$

$$a^{\alpha\beta}(x) := (1/2)\sigma^{\alpha\beta}(x)(\sigma^{\alpha\beta}(x))^*, \qquad D_i = \partial/\partial x^i, \qquad D_{ij} = D_i D_j.$$

We will assume that σ and b are uniformly Lipschitz with respect to x, $\sigma\sigma^*$ is uniformly nondegenerate, and c and f are uniformly bounded. In such a situation uniqueness of continuous viscosity solutions or even continuous L_p viscosity solutions of (1.2) is shown in [5] and therefore the fact of the independence of v of the probability space seems to be obvious.

Roughly speaking, the goal of this paper is to show that the value function does not change even if we keep the same space $(\Omega, \mathcal{F})$ but introduce probability measures by means of Girsanov's transformation *depending* on the policies of the players. We also show that the value function does not change if we allow the driving Wiener processes to depend on the policies of the players. Finally, we show that the value function does not change if we perform a random time change with the rate depending on the policies of the players.

These facts are well known in the theory of controlled diffusion processes and play there a very important role, in particular, while estimating the derivatives of the value function. A rather awkward substitute of them for stochastic differential games was used for the same purposes in [12]. Applying the results presented here one can make many constructions in [12] more natural and avoid introducing auxiliary "shadow" processes.

However, not all proofs in [12] can be simplified using our present methods. We deliberately avoided discussing the way to use the external parameters in contrast with [12] just to make the presentation more transparent.

Our proofs do not use anything from the theory of viscosity solutions and are based on a version of Święch's [14] idea as presented in [10] and a general solvability theorem in class $C^{1,1}$ of Isaacs equations from [9].

It is quite possible that Święch's approach from [14] based on inf sup convolutions can be further developed and used to prove our main results. We prefer using the above mentioned result from [9] for two reasons.

First, in previous articles we used this result to establish a W^2_p-solvability theorem for fully nonlinear equations with a "relaxed" convexity assumption, to showing that the L_p-viscosity solutions of Isaacs equations with VMO coefficients are in $C^{1+\kappa}$, and to establishing an algebraic rate of convergence of finite-difference approximations to solutions of Isaacs equations. Here we provide one more application of this result of [9].