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Maximums on trees

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Abstract

We study the minimal/endogenous solution R to the maximum recursion on weighted branching trees given by

$$R \stackrel{\mathcal{D}}{=} \left(\bigvee_{i=1}^{N} C_i R_i \right) \vee Q,$$

where $(Q, N, C_1, C_2, \ldots)$ is a random vector with $N \in \mathbb{N} \cup \{\infty\}$, $P(|Q| > 0) > 0$ and nonnegative weights $\{C_i\}$, and $\{R_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. copies of R independent of $(Q, N, C_1, C_2, \ldots)$; $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Furthermore, when $Q > 0$ this recursion can be transformed into its additive equivalent, which corresponds to the maximum of a branching random walk and is also known as a high-order Lindley equation. We show that, under natural conditions, the asymptotic behavior of R is power-law, i.e., $P(|R| > x) \sim H x^{-\alpha}$, for some $\alpha > 0$ and $H > 0$. This has direct implications for the tail behavior of other well known branching recursions.

MSC: 60H25; 60J80; 60F10; 60K05

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1. Introduction

In recent years considerable attention [12–14,3,4,16,2,7] has been given to the characterization and analysis of the solutions to the non homogeneous linear equation

$$R_L \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_{L,i} + Q, \tag{1}$$

where $(Q, N, C_1, C_2, \ldots)$ is a real-valued random vector with $N \in \mathbb{N} \cup \{\infty\}$, $P(|Q| > 0) > 0$, and $\{R_{L,i}\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(Q, N, C_1, C_2, \ldots)$ having the same distribution as R_L. Eq. (1) has applications in a wide variety of fields, including the analysis of divide and conquer algorithms [18,17], e.g. Quicksort [9]; the analysis of the PageRank algorithm [19,12]; and kinetic gas theory [7]. Our work in [13,14] shows that the so-called endogenous solution, as termed in [1], of (1), under the natural main root condition $E\left[\sum_{i=1}^{N} |C_i|^\alpha\right] = 1$ with positive derivative $0 < E\left[\sum_{i=1}^{N} |C_i|^\alpha \log |C_i|\right] < \infty$ for some $\alpha > 0$, has the power tail behavior,

$$P(|R_L| > t) \sim H_L t^{-\alpha}, \quad t \to \infty,$$

where $0 \le H_L < \infty$. The main tool used in deriving this result was a generalization of Goldie's Implicit Renewal Theorem [10] to weighted branching trees.

Motivated by a different set of applications, we study in this paper the maximum recursion on trees given by

$$R \stackrel{\mathcal{D}}{=} \left(\bigvee_{i=1}^{N} C_i R_i \right) \vee Q, \tag{2}$$

where $(Q, N, C_1, C_2, \ldots)$ is a random vector with $N \in \mathbb{N} \cup \{\infty\}$, nonnegative weights $\{C_i\}$, and $P(|Q| > 0) > 0$, and $\{R_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(Q, N, C_1, C_2, \ldots)$ having the same distribution as R. Here and throughout the paper we use $x \vee y$ and $x \wedge y$ to denote the maximum and minimum, respectively, of x and y. We point out that by taking the logarithm in (2) when $Q > 0$ a.s., we obtain the additive equivalent

$$X \stackrel{\mathcal{D}}{=} \bigvee_{i=1}^{N} (Y_i + X_i) \vee V, \tag{3}$$

where $X = \log R$, $Y_i = \log C_i$, $V = \log Q$, and the $\{X_i\}_{i \in \mathbb{N}}$ are i.i.d. copies of X, independent of $(V, N, Y_1, Y_2, \ldots)$. Note that for $N \equiv 1$ and $V \equiv 0$, (3) reduces to the classical Lindley's equation, satisfied by the reflected random walk; and when $V \not\equiv 0$, the recursion corresponds to a random walk reflected on a random barrier. In general, the preceding additive equation has been studied in the literature of branching random walks (see [1, Section 4.2]). Recursion (3) was termed "high-order Lindley equation" and studied in the context of queues with synchronization in [15]. Unlike the classical Lindley equation, it was shown in [15] that (3) can have multiple solutions. A more complete analysis of the existence and the characterization of the entire family of solutions was carried out in [6] (e.g., see Theorem 1 in [6]). In addition, it can be shown that the study of (3) arises in the context of today's massively parallel computing, e.g., consider a job that is split into smaller pieces which are sent randomly to different processors, and these pieces need to be synchronized in order to complete their processing. In addition to these applications, a