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Quadratic covariation estimates in non-smooth stochastic calculus

Sergio Angel Almada Monter

Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27599, United States

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Abstract

Given a Brownian Motion W, in this paper we study the asymptotic behavior, as $\varepsilon \to 0$, of the quadratic covariation between $f(\varepsilon W)$ and W in the case in which f is not smooth. Among the main features discovered is that the speed of the decay in the case $f \in C^\alpha$ is at least polynomial in ε and not exponential as expected. We use a recent representation as a backward–forward Itô integral of $[f(\varepsilon W), W]$ to prove an ε-dependent approximation scheme which is of independent interest. We get the result by providing estimates to this approximation. The results are then adapted and applied to generalize the results of Almada Monter and Bakhtin (2011) and Bakhtin (2011) related to the small noise exit from a domain problem for the saddle case.

Keywords: Non-smooth Itô's formula; Quadratic variation; Large deviation

1. Introduction

One of the central results of stochastic calculus is Itô's change of variables formula for twice differentiable transformations of semimartingales. It was realized recently that one also needs to study nonlinear maps that are not smooth enough to allow an application of the classical Itô formula. Various approaches to less regular changes of variables have been introduced, see [4,5,8,10,19], and references therein. These studies show that the key feature of the Itô formula, the quadratic covariation term, is well-defined under much weaker assumptions than those leading to the traditional formula. However, no nontrivial quantitative estimates of the arising quadratic covariation processes have appeared in the literature, to the best of our knowledge.

E-mail addresses: saalm56@gmail.com, salmada3@email.unc.edu.

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One area where such estimates are naturally needed is small random perturbations of dynamical systems. Often, in the course of a study of a stochastic system one has to make a simplifying change of coordinates, transforming the system locally to a simpler one. If the transformation map is C^2, then one can apply the classical Itô calculus and easily control the Itô correction term. However, there are situations where a natural change of variables is less regular than C^2, and in these cases there is no readily available tool that could be used to control the generalized Itô correction.

The goal of this paper is to close this gap and provide quantitative estimates on the generalized Itô correction term under nonclassical assumptions on the transformation.

Let us now be more precise. Let W be a standard 1-dimensional Wiener process on a complete probability space $(\Omega, \Sigma, \mathbf{P})$ and $\varepsilon > 0$ be a constant. If $g : \mathbb{R} \to \mathbb{R}$ is C^2, then the classical Itô formula is (see [17, Section II.7])

$$g(\varepsilon W(t)) - g(0) = \varepsilon \int_0^t g'(\varepsilon W(s)) dW(s) + \frac{\varepsilon^2}{2} \int_0^t g''(\varepsilon W(s)) ds.$$

Introducing $f = g' \in C^1$, we can also rewrite the second term in the r.h.s. as quadratic covariation between $f(\varepsilon W)$ and εW: for $Q_\varepsilon(t) = [f(\varepsilon W), \varepsilon W](t)$, we have

$$Q_\varepsilon(t) = \varepsilon^2 \int_0^t f'(\varepsilon W(s)) ds, \quad t \geq 0. \tag{1}$$

In particular, for any $T > 0$, $\varepsilon^{-1} \sup_{t \leq T} Q_\varepsilon(t) \to 0$ in probability as $\varepsilon \to 0$. In this paper we show that this convergence holds in the case in which f is not differentiable.

The motivation for this problem relies on small random perturbation of dynamical systems. Suppose that b is a vector field with a critical point at x^* and let S denote the flow generated by b:

$$\frac{d}{dt} S^t x = b(S^t x), \qquad S^0 x = x.$$

It is well known (see Section 2.8 of [16]) that there is a continuous change of variables g so that locally around $g(x^*)$ the flow $g(S^t x)$ behaves like the linearized version of S. In the small random perturbation case, this combined with the traditional Itô formula imply (see, e.g. [2,14]) that if g is at least C^2, then the system

$$dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \varepsilon dW(t), \qquad X_\varepsilon(0) = x_0,$$

could be analyzed by working with the linear system

$$d\tilde{X}_\varepsilon(t) = \left(A\tilde{X}_\varepsilon(t) + \frac{\varepsilon^2}{2} \Phi_\varepsilon(X^\varepsilon(t)) \right) dt + \varepsilon \sigma(X_\varepsilon(t)) dW(t), \qquad \tilde{X}_\varepsilon(0) = g(x_0),$$

where x_0 is close enough to x^*, A is the Jacobian of b at x^*, σ is at least a continuous matrix valued function, and $\varepsilon^2 \Phi_\varepsilon$ is the term corresponding to the quadratic covariation between $g'(X_\varepsilon)$ and X_ε. There are well established cases for which g is known to be C^1, see e.g. Hartman Theorem on Section 2.8 of [16]. In these cases, an already known C^1 formulation of Itô's formula implies a similar analogy between the non-linear and linear systems. Hence, estimates that show that in these cases the quadratic covariation term decays faster than the Itô term allow to reduce the local analysis to a simpler exit problem for Ornstein–Uhlenbeck processes.

The analysis of the quadratic covariation $[g'(X), X]$ in connection with extensions of Itô's formula for functions $g \notin C^2$ is fundamental for nonsmooth Itô calculus, see [7,9,10,19,18].