Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 125 (2015) 362–370

stochastic processes and their applications

www.elsevier.com/locate/spa

Phase transition for finite-speed detection among moving particles

Vladas Sidoravicius[a], Alexandre Stauffer[b,*]

[a] *Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil*
[b] *Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK*

Received 15 January 2014; received in revised form 10 July 2014; accepted 8 September 2014
Available online 17 September 2014

Abstract

Consider the model where particles are initially distributed on $\mathbb{Z}^d$, $d \geq 2$, according to a Poisson point process of intensity $\lambda > 0$, and are moving in continuous time as independent simple symmetric random walks. We study the escape versus detection problem, in which the target, initially placed at the origin of $\mathbb{Z}^d$, $d \geq 2$, and changing its location on the lattice in time according to some rule, is said to be detected if at some finite time its position coincides with the position of a particle. For any given $S > 0$, we consider the case where the target can move with speed at most S, according to any continuous function and can adapt its motion based on the location of the particles. We show that, for any $S > 0$, there exists a sufficiently small $\lambda_* > 0$, so that if the initial density of particles $\lambda < \lambda_*$, then the target can avoid detection forever.

MSC: 82C43; 60G55; 60K35

Keywords: Poisson point process; Target detection; Oriented space–time percolation

1. Introduction

Let Π be a Poisson point process of intensity $\lambda > 0$ on $\mathbb{Z}^d$, $d \geq 2$. We label all points of this process by positive integers in some arbitrary way, *i.e.* $\Pi = \{p_j\}_{j \geq 1}$, and interpret the points of

* Corresponding author.
E-mail addresses: vladas@impa.br (V. Sidoravicius), a.stauffer@bath.ac.uk (A. Stauffer).

http://dx.doi.org/10.1016/j.spa.2014.09.007
0304-4149/

Π as *particles*. We denote by $\eta_j(0),\ j \geq 1$, the initial position of the jth particle, and we will assume that each particle $p_i,\ i \geq 1$, moves as an independent continuous-time random walk on $\mathbb{Z}^d$. More formally, for each $k \geq 1$, let $(\zeta_k(t))_{t\geq 0}$ be an independent continuous-time random walk on $\mathbb{Z}^d$ starting from the origin. Then $\eta_k(t) := \eta_k(0) + \zeta_k(t)$ denotes the location of the kth particle at time t.

In addition, we consider an extra particle, called *target*, which at time 0 is positioned at the origin, and is moving on $\mathbb{Z}^d$, $d \geq 2$ in time, according to a certain prescribed rule. We say that the target is *detected* at time t, if there exists a particle p_j located at time t at the same vertex as the target. We will assume that the target particle wants to evade detection and can do so by moving in continuous time by means of nearest-neighbor jumps on $\mathbb{Z}^d$, which can depend on the past, present and future positions of the particles.

More precisely, let $\mathcal{P}$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{Z}^d$ for which $g(0) = 0$ and such that the following holds:

$$\begin{aligned}&\text{for any } g \in \mathcal{P},\ \text{any } t \geq 0 \text{ and any } \xi > 0,\ \text{if } \|g(t+\xi) - g(t)\| > 1 \text{ then there}\\ &\quad \text{exists } \xi' \in (0,\xi) \text{ for which } 0 < \|g(t+\xi') - g(t)\| < \|g(t+\xi) - g(t)\|. \end{aligned} \tag{1}$$

We view $\mathcal{P}$ as the set of all permitted trajectories for the target, and $g(t),\ g \in \mathcal{P}$, denotes the position of the target at time t. Condition (1) in the definition of $\mathcal{P}$ prevents the target to make long range jumps, i.e. for any trajectory $g \in \mathcal{P}$, the target is allowed to jump only between nearest neighbor vertices of $\mathbb{Z}^d$.

We say that $g \in \mathcal{P}$ is *detected* at time t if there exists a particle $p_j \in \Pi$, for some $j \geq 1$, such that $\eta_j(t) = g(t)$, and define the detection time of g as follows:

$$T_{det}(g) = \inf\Big\{t \geq 0\colon g(t) \in \bigcup_{k\geq 1} \eta_k(t)\Big\}.$$

In [9, Theorem 1.1] it was shown that there exists a phase transition in λ so that, if λ is large enough, $\mathbf{P}(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}) = 1$. Hence, the target cannot avoid detection forever even if it knew the past, present and *future* positions of the particles at all times, and could move at any time at any arbitrarily large speed.

Here we consider a parameter $0 < S < +\infty$ and let $\mathcal{P}_S \subset \mathcal{P}$ be the set of all trajectories $g \in \mathcal{P}$ with maximum *speed* S, *i.e.*,

$$\mathcal{P}_S := \{g \in \mathcal{P} : \forall t \geq 0\ \forall \xi > 0,\ \|g(t+\xi) - g(t)\| \leq \xi S \vee 1\}.$$

Then define

$$\lambda_{det}(S) = \inf\Big\{\lambda \geq 0\colon \mathbf{P}(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}_S) = 1\Big\}$$

and

$$\lambda_{det}(\infty) = \inf\Big\{\lambda \geq 0\colon \mathbf{P}(T_{det}(g) < \infty \text{ for all } g \in \mathcal{P}) = 1\Big\}.$$

The main result in [9, Theorem 1.1], mentioned above, gives that $\lambda_{det}(\infty) \in (0,\infty)$. Since for any $S \leq S'$ we have $\mathcal{P}_S \subseteq \mathcal{P}_{S'}$, then

$$\lambda_{det}(S) \leq \lambda_{det}(S') \leq \lambda_{det}(\infty) < \infty.$$

It was also observed in [9], that for sufficiently small $\lambda > 0$, there is a strictly positive probability for the target, starting from the origin, to avoid detection forever, provided it can move at any time at any arbitrarily large speed, *i.e.* $\lambda_{det}(\infty) > 0$.