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Annealed asymptotics for the parabolic Anderson model with a moving catalyst

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Abstract

This paper deals with the solution u to the parabolic Anderson equation $\partial u/\partial t = \kappa\Delta u + \xi u$ on the lattice $\mathbb{Z}^d$. We consider the case where the potential ξ is time-dependent and has the form $\xi(t,x) = \delta_0(x - Y_t)$ with Y_t being a simple random walk with jump rate $2d\varrho$. The solution u may be interpreted as the concentration of a *reactant* under the influence of a single *catalyst* particle Y_t.

In the first part of the paper we show that the moment Lyapunov exponents coincide with the upper boundary of the spectrum of certain Hamiltonians. In the second part we study intermittency in terms of the moment Lyapunov exponents as a function of the model parameters κ and ϱ.

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1. Introduction

1.1. The parabolic Anderson problem and its interpretation

The main object of our investigation is the solution $u\colon \mathbb{R}^+ \times \mathbb{Z}^d \to \mathbb{R}^+$ to the Cauchy problem for the heat equation with a random time-dependent potential:

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$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \kappa \Delta u(t,x) + \xi(t,x)\, u(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \\ u(0,x) = 1, & x \in \mathbb{Z}^d. \end{cases} \tag{1.1}$$

Here, $\kappa \in \mathbb{R}^+$ is a diffusion constant and Δ is the discrete Laplacian acting on $f : \mathbb{Z}^d \to \mathbb{R}$ as

$$\Delta f(x) = \sum_{y \sim x} [f(y) - f(x)],$$

while

$$\xi(t) = \left\{ \xi(t,x) \mid x \in \mathbb{Z}^d \right\}, \quad t \in \mathbb{R}^+,$$

is an $\mathbb{R}$-valued random field evolving over time that "drives" the equation. Problem (1.1) is referred to as the *parabolic Anderson model*. It is the parabolic analogue of the Schrödinger equation with a time-dependent random potential.

A popular heuristic interpretation of the model arises from population dynamics. In this context the function $u(t,x)$ describes the mean number of particles present at x at time t when starting with one particle per site. Particles perform independent random walks on $\mathbb{Z}^d$ with jump rate $2d\kappa$ and split into two at rate ξ if $\xi > 0$ (source) or die at rate $-\xi$ if $\xi < 0$ (sink).

If ξ is a nonnegative field, then we can interpret the problem in (1.1) also as a linearized model of chemical reactions. In this case, the solution of the equation describes the evolution of reactant particles under the influence of a catalyst medium ξ. More precisely, u describes the expected number of reactant particles if its time evolution is governed by the following rules:

(i) at time $t = 0$, each lattice site is occupied by one reactant;
(ii) reactants act independently of each other;
(iii) a reactant at x jumps to a neighboring site y at rate κ;
(iv) a reactant at x splits into two at rate $\xi(t,x)$.

Another example is mathematical modeling in evolution theory. Considering a fixed size population, one may describe its evolution by the Fisher–Eigen equation of population genetics which is a version of (1.1). Hereby $\mathbb{Z}^d$ represents the space of phenotypes, Δ describes mutation and ξ is the fitness. See e.g. Ebeling et al. [5, Sect. 2] for such an approach.

Characteristically for the parabolic Anderson model, the two terms on the right hand side of Eq. (1.1) compete with each other. The diffusion induced by Δ tends to make u flat whereas ξ tends to make u bumpy. In the context of population dynamics, there is a competition between individuals spreading out by diffusion and clumping around sources.

Studying problem (1.1), we distinguish between the *quenched* setting which describes the almost sure behaviour of u conditioned on ξ, and the *annealed* setting, where we average over ξ. The present paper deals with the annealed setting.

The theory currently available for the model covers various forms of the potential ξ. In the present paper we consider the case where ξ has the form

$$\xi(t,x) = \delta_{Y_t}(x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{Z}^d, \tag{1.2}$$

where $(Y_t)_{t \geq 0}$ is a random walk with generator $\varrho\Delta$ starting at the origin and $\delta_y(x)$ is the Kronecker symbol. The corresponding expectation will be denoted by $\langle \cdot \rangle$. The parameter $\varrho \in [0, \infty)$ is the diffusion constant of the catalyst. In the context of chemical reactions, we can interpret ξ as the reaction rate induced by a single catalyst particle, which performs a random walk in $\mathbb{Z}^d$ with jump rate $2d\varrho$. Reactants split into two at rate 1 if they are at the same lattice