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Verification theorems for stochastic optimal control problems via a time dependent Fukushima–Dirichlet decomposition

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Abstract

This paper is devoted to presenting a method of proving verification theorems for stochastic optimal control of finite dimensional diffusion processes without control in the diffusion term. The value function is assumed to be continuous in time and once differentiable in the space variable ($C^{0,1}$) instead of once differentiable in time and twice in space ($C^{1,2}$), like in the classical results. The results are obtained using a time dependent Fukushima–Dirichlet decomposition proved in a companion paper by the same authors using stochastic calculus via regularization. Applications, examples and a comparison with other similar results are also given.

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1. Introduction

In this paper we want to present a method for obtaining verification theorems for stochastic optimal control problems of finite dimensional diffusion processes without control in the

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diffusion term. The method is based on a generalized Fukushima–Dirichlet decomposition proved in the companion paper [22]. Since this Fukushima–Dirichlet decomposition holds for functions $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ that are C^0 in time and C^1 in space ($C^{0,1}$ in symbols), our verification theorem has the advantage of requiring less regularity of the value function V than the classical ones which need C^1 regularity in time and C^2 in space of V ($C^{1,2}$ in symbols), see e.g. [13, pp. 140,163,172].

There are also other verification theorems that work in cases when the value function is nonsmooth: e.g. it is possible to prove a verification theorem in the case when V is only continuous (see [23,27], [39, Section 5.2]) in the framework of viscosity solutions. However all these results applied to our cases are weaker than ours, for reasons that are clarified in Section 8.

Since the method is relatively complex we present first in Section 2 the statement of our verification Theorems 2.7 and 2.8 in a model case with simplified assumptions (which substantially yield nondegeneracy of the diffusion coefficient). In the same section we also include Section 2.1 where we give some comments on the theorem, its applicability and its relationship with other similar results.

Below we pass to the body of the paper giving first some notation in Section 3 and then presenting the general statements (including also possible degeneracy of the diffusion coefficient) and their proof in Section 4. Section 5 is devoted to necessary conditions and optimal feedbacks.

Sections 6 and 7 contain applications of our technique to more specific classes of problems where other techniques are more difficult to use. The first is a case of an exit time problem where the HJB equation is nondegenerate but $C^{1,2}$ regularity is not known to hold due to the lack of regularity of the coefficients; the second is a case where the HJB equation is degenerate parabolic. Finally in Section 8 we compare our results with those of other verification techniques.

2. The statement of the verification theorems in a model case

To clarify our results we describe briefly and informally below the framework and the statement of the verification theorem that we are going to prove in a model case. The precise statements and proofs are given in Section 4; then in Sections 6 and 7 applications to more specific (and somewhat more difficult) classes of problems are given. We decided on this structure since it is difficult to provide a single general result: we can say that we introduce a technique, based on the Fukushima–Dirichlet decomposition and on its representation given in [22], that can be adapted with some work to different settings each time with a different adaptation.

First we take a given stochastic basis $(\Omega, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$ that satisfies the so-called usual conditions, a finite dimensional Hilbert space $A = \mathbb{R}^n$ (the state space), a finite dimensional Hilbert space $E = \mathbb{R}^m$ (the noise space), a set $U \subseteq \mathbb{R}^k$ (the control space). We fix then a terminal time $T \in [0, +\infty]$ (the horizon of the problem, that can be finite or infinite but is fixed), an initial time and state $(t, x) \in [0, T] \times A$ (which will vary as usual in the dynamic programming approach). The state equation is (recall that $\mathcal{T}_t = [t, T] \cap \mathbb{R}$)

$$\begin{aligned} dy(s) &= [F_0(s, y(s)) + F_1(s, y(s), z(s))]\, ds + B(s, y(s))\, dW(s), \quad s \in \mathcal{T}_t, \\ y(t) &= x, \end{aligned} \tag{1}$$

where the following holds (for a matrix B, by $\|B\|$ we mean $\sum_{i,j} |b_{ij}|$ and, given E, F finite dimensional spaces, by $\mathcal{L}(E, F)$, we mean the set of linear operators from E to F).

Hypothesis 2.1. 1. $z : \mathcal{T}_t \times \Omega \to U$ (the control process) is measurable, locally integrable in t for a.e. $\omega \in \Omega$, adapted to the filtration $(\mathcal{F}_s)_{s\geq 0}$.