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Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions

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Abstract

In the present paper, we characterize the behavior of supercritical branching processes in random environment with linear fractional offspring distributions, conditioned on having small, but positive values at some large generation. As it has been noticed in previous works, there is a phase transition in the behavior of the process. Here, we examine the strongly and intermediately supercritical regimes The main result is a conditional limit theorem for the rescaled associated random walk in the intermediately case.

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Keywords: Branching process in random environment; Supercritical; Conditional limit theorem

1. Introduction

Branching processes in random environment (BPRE) are a stochastic model for the development of a population in discrete time. The model has first been introduced in [8,21]. In generalization to Galton–Watson processes, the reproductive success of all individuals of

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a generation is influenced by an environment which varies in an independent fashion from generation to generation.

As first noted in [1,13], there is a phase transition in the behavior of subcritical BPRE (see e.g. for an overview [12] and for detailed results [2–6,17,22–25,14]. Only recently, there has been interest in a phase transition in supercritical processes, conditioned on surviving and having small values at some large generation (see [9,10,20]). For the scaling limit of supercritical branching diffusions, a phase transition has been noted in [18].

In [9,18], the terminology of strongly, intermediately and weakly supercritical BPRE has been introduced in analogy to subcritical BPRE. In the present paper, we focus on the phase transition from strongly to intermediately supercriticality and characterize these regimes with limit results.

Let us formally introduce a branching process in random environment $(Z_n)_{n\in\mathbb{N}_0}$. For this, let Q be a random variable taking values in Δ, the space of all probability distributions on $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. An infinite sequence $\Pi = (Q_1, Q_2, \ldots)$ of i.i.d. copies of Q is called a *random environment*. By Q_n, we denote the (random) offspring distribution of an individual at generation $n - 1$.

Let Z_n be the number of individuals in generation n. Then Z_n is the sum of Z_{n-1} independent random variables with distribution Q_n. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is then called a *branching process in the random environment* Π, if Z_0 is independent of Π and given Π the process $Z = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\big(Z_n \bigm| Z_{n-1} = z,\ \Pi = (q_1, q_2, \ldots)\big) \ = \ q_n^{*z} \tag{1.1}$$

for every $n \in \mathbb{N} = \{1, 2, \ldots\}$, $z \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \Delta$, where q^{*z} is the z-fold convolution of the measure q.

By $\mathbb{P}$, we will denote the corresponding probability measure on the underlying probability space. We shorten $Q(\{y\}), q(\{y\})$ to $Q(y), q(y)$ and write

$$\mathbb{P}(\cdot \mid Z_0 = z) =: \mathbb{P}_z(\cdot).$$

For convenience, we write $\mathbb{P}(\cdot)$ instead of $\mathbb{P}_1(\cdot)$. Throughout the paper, we assume that the offspring distributions have the following form,

$$q\big(0\big) = a, \qquad q\big(k\big) = \frac{(1-a)(1-p)}{p}\, p^k, \quad \text{for } k \geq 1$$

where $a \in [0, 1)$ and $p \in (0, 1)$ are two random parameters. Note that $a = 1$ would imply that an individual becomes extinct with probability one. Thus we exclude this case. This class of offspring distributions is often called *linear fractional* as the generating functions have an explicit formula as a quotient of two linear functions. Also the concatenation of two linear fractional functions is again linear fractional and thus the generating function of Z_n can be calculated explicitly in this case. An important tool in the analysis of BPRE is the associated random walk $S = (S_n)_{n\geq 0}$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1},\ n \geq 1$ defined by

$$X_n := \log m(Q_n),$$

where

$$m(q) := \sum_{y=0}^{\infty} y\, q(y)$$